Existence, stabilité et instabilité d'ondes stationnaires pour quelques équations de Klein-Gordon et Schrödinger non linéaires

Stefan Le Coz

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Existence, stabilité et instabilité d'ondes stationnaires pour quelques équations de Klein-Gordon et Schrödinger non linéaires

par

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Président	Charles STUART
Rapporteurs	Thierry CAZENAVE
	Thierry GALLAY
Examinateurs	Mariana HARAGUS
	Louis JEANJEAN
	Arnd SCHEEL

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Introduction

Cette thèse se divise en quatre chapitres correspondant à quatre publications ou projets de publication. Les trois premiers sont consacrés à des questions d'existence, de stabilité ou d'instabilité d'ondes stationnaires pour des équations de Schrödinger non linéaires, tandis que le quatrième traite d'instabilité pour des équations de Klein-Gordon non linéaires. Le premier chapitre correspond à une version plus détaillée d'un article co-signé avec Louis Jeanjean [18]. Le deuxième chapitre est une prépublication [24] dans laquelle les questions analytiques ont été traitées en collaboration avec Reika Fukuizumi ; quant aux résultats numériques, ils sont dus à Gadi Fibich et ses élèves Barush Ksherim et Yonatan Sivan. Je suis seul auteur du troisième chapitre [23] et le quatrième chapitre est le fruit d'un travail commun avec Louis Jeanjean [19].

Une équation de Schrödinger non linéaire est une équation de la forme

$$iu_t + \Delta u + f(x,u) = 0 \tag{1}$$

où $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ et $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ est une non-linéarité étendue à $\mathbb{R}^N \times \mathbb{C}$ en posant quel que soit $x \in \mathbb{R}^N$ $f(x,z) := f(x,|z|)z/|z|$ pour $z \in \mathbb{C} \setminus \{0\}$ et $f(x,0) = 0$.

Sous certaines conditions sur f, le problème de Cauchy pour (1) est localement bien posé dans $H^1(\mathbb{R}^N)$ (voir par exemple [6, chapitre 4]) et soit la solution du problème de Cauchy existe globalement, soit elle explose en temps fini (ce qu'on désigne sous le nom de *blow-up alternative*). De plus, si on définit l'*énergie* E et la *charge* Q pour $v \in H^1(\mathbb{R}^N)$ par

$$\begin{aligned} E(v) &:= \frac{1}{2}\|\nabla v\|_2^2 - \int_{\mathbb{R}^N} F(x,v)dx, \\ Q(v) &:= \|v\|_2^2, \end{aligned}$$

où $F(x,s) = \int_0^{|s|} f(x,\sigma)d\sigma$, alors ces deux quantités sont conservées au cours du temps.

Pour de nombreuses équations non linéaires dispersives, on observe dans certaines situations une compensation entre l'effet dispersif du laplacien et les effets non linéaires qui donne lieu à la génération d'*ondes solitaires*. Il s'agit de solutions de ces équations qui peuvent subir des modifications de phase ou des translations en espace mais dont le profil reste intact au cours du temps. Concrètement, la première observation d'une onde solitaire remonte à 1834 : John Scott Russell parcourt à cheval plusieurs kilomètres le long d'un canal pour observer la propagation à l'identique de l'onde créée par l'arrêt brusque d'une barge. Cependant, il faut attendre les travaux de Korteweg et de Vries en 1895 pour que le phénomène trouve une première justification théorique et ce n'est qu'après les années 1950 que l'étude des ondes solitaires prendra véritablement son essor. Depuis, les équations admettant des ondes solitaires ont connu un fort engouement aussi bien de la part des mathématiciens que des physiciens (voir par exemple [6, 9, 28, 33] pour une revue de questions physiques et mathématiques autour des ondes solitaires et pour une bibliographie détaillée).

Pour l'équation de Schrödinger, les ondes solitaires auxquelles nous nous intéressons sont les *ondes stationnaires*. Ce sont des solutions de (1) de la forme $e^{i\omega t}\varphi_\omega(x)$ avec $\omega \in \mathbb{R}$ et $\varphi_\omega \in H^1(\mathbb{R}^N)$ qui vérifie

$$-\Delta\varphi_\omega + \omega\varphi_\omega - f(x,\varphi_\omega) = 0. \tag{2}$$

La première étude mathématique de l'existence de solutions de (2) en dimension supérieure à 3 remonte à un article de Strauss [32] en 1977. Lorsque la non-linéarité f est autonome (i.e. $f(x,s) \equiv f(s)$), Berestycki et Lions [4] ont donné en 1983 des conditions quasi-optimales garantissant l'existence de solutions dans $H^1(\mathbb{R}^N)$ pour (2) lorsque $N \geqslant 3$ et $N = 1$. Le cas $N = 2$ fut traité peu de temps après par Berestycki, Gallouet et Kavian [3]. En particulier, si on définit la fonctionnelle naturellement associée à (2) par

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \int_{\mathbb{R}^N} F(v)dx$$

alors sous les hypothèses de [3, 4] il existe des solutions φ vérifiant

$$S(\varphi) = m := \inf\{S(v)\big| v \in H^1(\mathbb{R}^N)\setminus\{0\} \text{ est une solution de (2)}\}.$$

Ces solutions sont dites *de plus petite énergie*, ou *états fondamentaux*, et m est le *niveau de plus petite énergie*.

Lorsque f est non-autonome, seuls des résultats partiels sont connus. Dans le premier chapitre de cette thèse, on prouve un résultat d'existence pour (2) lorsque la non-linéarité f est de la forme $f(x,s) = V(x)g(s)$. Ici, V désigne un potentiel réel et g une non-linéarité vérifiant

(H1) V se comporte comme $|x|^{-b}$ à l'infini avec $0 < b < 2$,
(H2) g se comporte comme s^p en 0 avec $1 < p < 1 + (4-2b)/(N-2)$.

(voir le chapitre 1 pour un énoncé précis des hypothèses)

Théorème 1. *Pour une non-linéarité de la forme $V(x)g(s)$ vérifiant (H1)-(H2), il existe $\omega_0 > 0$ tel que (2) admet une solution non-triviale φ_ω pour tout $\omega \in (0, \omega_0)$.*

Lors de la recherche de solutions pour des problèmes non-autonomes du type (2), une des difficultés majeures auxquelles on est confronté est l'absence d'estimations a priori sur les suites de Palais-Smale. De fait, la majorité des travaux sur le sujet se restreignent à des situations où la non-linéarité g satisfait des hypothèses fortes du type condition de superquadraticité d'Ambrosetti et Rabinowitz. Dans notre cas, nous surmontons cette difficulté en nous inspirant d'une méthode introduite par Berti et Bolle [5] en 2003 dans le contexte de l'équation des ondes. On cherche à obtenir les solutions de (2) comme points critiques, au niveau du col, de la fonctionnelle associée à (2)

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \int_{\mathbb{R}^N} V(x)G(v)dx,$$

où $G(s) := \int_0^{|s|} g(\sigma)d\sigma$. Cependant, s'il est vraisemblable que la fonctionnelle S admet une géométrie de col, montrer directement que les suites de Palais-Smale sont bornées semble hors de portée sous nos faibles hypothèses sur g. Pour surmonter cette difficulté, notre méthode consiste à tronquer convenablement la fonctionnelle S à l'extérieur d'une boule de $H^1(\mathbb{R}^N)$. On montre alors que la fonctionnelle tronquée a une géométrie de col et que ses suites de Palais-Smale au niveau du col sont à l'intérieur de la boule où la fonctionnelle d'origine et la fonctionnelle tronquée coïncident. Montrer la convergence des suites de Palais-Smale permet alors d'obtenir un point critique de S, donc une solution de (2).

Une fois leur existence établie, l'une des questions majeures dans l'étude des ondes solitaires est leur stabilité ou leur instabilité. Déjà dans son mémoire de 1844 [29], Russell mentionnait les remarquables propriétés de stabilité des ondes solitaires qu'il avait pu observer. Néanmoins, le développement d'une théorie mathématique rigoureuse de la stabilité ne commence qu'en 1972 avec les travaux de Benjamin [1] sur l'équation de Korteweg-de Vries. La stabilité étudiée par Benjamin est dite *orbitale*, c'est également ce type de stabilité que nous considérons dans le cadre de cette thèse.

L'orbite d'une onde stationnaire est déterminée par les propriétés de symétrie de l'équation. Par exemple, dans le cas où la non-linéarité est de type puissance

$$iu_t + \Delta u + |u|^{p-1}u = 0 \tag{3}$$

et si φ est une solution de

$$-\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \tag{4}$$

alors $e^{i\theta}\varphi(x-y)$ est également une solution de (4) quelque soit $\theta \in \mathbb{R}$ et $y \in \mathbb{R}^N$. Dans cette situation, l'orbite d'une onde stationnaire $u(t,x) = e^{i\omega t}\varphi_\omega(x)$ est l'ensemble

$$\mathcal{O}(\varphi_\omega) = \{e^{i\theta}\varphi_\omega(\cdot - y), \theta \in \mathbb{R}, y \in \mathbb{R}^N\}.$$

Pour H un espace de fonctions (en pratique $H^1(\mathbb{R}^N)$ ou son sous-espace des fonctions radiales $H^1_{\mathrm{rad}}(\mathbb{R}^N)$), on définit la stabilité orbitale dans H de l'onde $e^{i\omega t}\varphi_\omega(x)$ de la façon suivante. Pour tout $\varepsilon > 0$ il existe $\delta > 0$ tel que pour tout $u_0 \in H$ vérifiant $||\varphi_\omega - u_0||_H < \delta$ on a

$$\sup_{t\in[0,+\infty)} \inf_{v\in\mathcal{O}(\varphi_\omega)} ||v - u(t)||_H < \varepsilon,$$

où $u(t)$ est la solution de (3) associée à u_0.

Pour l'équation (3), Cazenave et Lions [7] ont montré en 1982 que les ondes stationnaires associées aux états fondamentaux de (4) sont stables dans $H^1(\mathbb{R}^N)$ si $1 < p < 1 + \frac{4}{N}$. Leur approche repose sur le fait que les états fondamentaux peuvent, dans ce cas, être caractérisés comme des minimiseurs de S sur une sphère de $L^2(\mathbb{R}^N)$. Leur résultat est optimal, dans la mesure où l'onde stationnaire est instable si $1 + \frac{4}{N} \leqslant p < 1 + \frac{4}{N-2}$ (avec $\frac{4}{N-2} = +\infty$ si $N = 1, 2$), voir [2, 34]. Cette approche s'est avérée efficace dans de nombreuses situations. Cependant, elle présente deux inconvénients. D'une part, la stabilité obtenue par cette approche correspond à une notion de stabilité potentiellement plus faible que celle de stabilité orbitale. En effet, ce qu'on montre par cette méthode est la stabilité de l'ensemble des états fondamentaux ; or cet ensemble ne coïncide avec l'orbite de l'onde stationnaire que s'il y a unicité de l'état fondamental aux symétries de l'équation près. D'autre part, cette approche est intimement liée aux états fondamentaux et ne permet pas de traiter d'autres états. En particulier, les solutions obtenues dans le Théorème 1 ne sont ni forcément uniques, ni caractérisées comme des minimiseurs de S, et on ne peut pas recourir à l'approche de Cazenave et Lions pour étudier leur stabilité.

À la même période, en 1985, Shatah et Strauss [31] ont introduit une méthode permettant d'étudier la stabilité et l'instabilité des équations non linéaires de Schrödinger et Klein-Gordon. Ils ont ensuite développé cette méthode en collaboration avec Grillakis [14, 15] pour traiter de systèmes hamiltoniens très généraux. Dans le cas de (1), cette théorie permet de déterminer si l'onde stationnaire $e^{i\omega t}\varphi_\omega(x)$ est stable ou instable en fonction de deux critères :

(critère spectral)	nombre de valeurs propres négatives de $S''(\varphi_\omega)$,
(critère de pente)	signe de $\frac{\partial}{\partial\omega}\|\varphi_\omega\|_2^2$.

Cette théorie de Grillakis, Shatah et Strauss se révèle être très efficace dans des situations où on connaît explicitement la dépendance de la famille (φ_ω) dans le paramètre ω. C'est notamment le cas lorsque la non-linéarité est de type puissance, éventuellement avec une dépendance en espace « simple », par exemple, lorsque $f(x,s) = |x|^{-b}|s|^{p-1}s$.

Cependant, dès que la dépendance de la famille (φ_ω) dans le paramètre ω n'est plus explicite, cette théorie devient très difficile à mettre en œuvre. De ce point de vue, la situation du Théorème 1 est très défavorable, car la dépendance dans le paramètre ω n'est même pas nécessairement continue. Malgré tout, il est possible de dériver des travaux de Grillakis, Shatah et Strauss un critère de stabilité basé sur une forme de coercitivité pour $S''(\varphi_\omega)$. Ce critère sera plus difficile à vérifier en pratique, mais vaudra dans des situations où on ne peut pas obtenir le critère de pente. Plus précisément, si pour $v \in H^1(\mathbb{R}^N)$ telle que $(v, \varphi_\omega)_2 = (v, i\varphi_\omega)_2 = 0$ on a

$$\textit{(critère de coercivité)} \qquad \langle S''(\varphi_\omega)v, v\rangle \geqslant C\|v\|^2_{H^1(\mathbb{R}^N)},$$

avec $C > 0$ indépendant de v, alors l'onde stationnaire $e^{i\omega t}\varphi_\omega(x)$ est stable dans $H^1(\mathbb{R}^N)$. Dans le chapitre 1, on exploite ce critère pour prouver la stabilité des solutions du Théorème 1.

Théorème 2. *On suppose que la non-linéarité est de la forme $V(x)g(s)$ et vérifie (H1)-(H2) avec $1 < p < \frac{4-2b}{N}$. Alors il existe $0 < \omega_1 \leqslant \omega_0$ tel que pour $\omega \in (0, \omega_1)$ les ondes stationnaires $e^{i\omega t}\varphi_\omega(x)$ obtenues dans le Théorème 1 sont stables dans $H^1(\mathbb{R}^N)$.*

Notre point de départ pour prouver le Théorème 2 est le travail de de Bouard et Fukuizumi [8] en 2005. Dans cet article, les auteurs étudient le même type d'équations en se restreignant à des non-linéarités de type puissance et sous des hypothèses plus fortes sur le potentiel V. Le plan d'étude de la stabilité est le suivant. Tout d'abord, on montre un résultat de convergence des solutions obtenues dans le Théorème 1 vers l'unique solution positive ψ du problème limite

$$-\Delta\psi + \psi - \frac{1}{|x|^b}|\psi|^{p-1}\psi = 0.$$

Puis, à l'aide d'une étude spectrale, on montre le critère de coercivité pour le problème limite. La partie difficile de cette étude spectrale consiste à prouver un résultat de non-dégénérescence de l'opérateur $S''(\psi)$. Bien que ce résultat soit déjà énoncé dans [8], la preuve qui y est donnée comporte plusieurs lacunes. On donne dans le chapitre 1 une preuve complète de ce résultat de non-dégénérescence. On conclut en montrant que le critère de coercivité est vérifié pour ω petit.

Les méthodes développées dans le chapitre 1 de cette thèse ont été employées avec succès par Kikuchi [22] dans le contexte de l'équation de Schrödinger-Poisson-Slater pour prouver un résultat d'existence et de stabilité d'ondes stationnaires. D'autre part, Genoud et Stuart [12] ont également abordé des questions d'existence et de stabilité pour des problèmes du type (1) avec une non-linéarité de la forme $V(x)|s|^{p-1}s$. Sous des hypothèses plus fortes sur V, ils obtiennent l'existence de solutions par une méthode de bifurcation et étudient leur stabilité ou instabilité.

Le deuxième chapitre de cette thèse traite de la stabilité et de l'instabilité des ondes stationnaires de l'équation

$$i\partial_t u + \partial_{xx} u + \gamma u\delta + |u|^{p-1}u = 0, \tag{5}$$

où $x \in \mathbb{R}$, δ désigne la distribution de Dirac à l'origine et γ un paramètre réel. Ce type d'équation intervient notamment en optique non linéaire ou dans la modélisation de brins d'ADN comportant certains défauts. Si cette équation est utilisée des physiciens depuis les années 1990, la première étude mathématique rigoureuse semble due à Goodman, Holmes et Weinstein [13] et date de 2004. Même si la question des ondes stationnaires et de leur stabilité est évoquée dans cette étude, les auteurs se concentrent surtout sur l'impact de la masse de Dirac sur l'évolution de la solution de l'équation lorsque la donnée initiale est un état fondamental de l'équation non perturbée localisé loin de 0. Plusieurs autres études ont été réalisées dans le même esprit, notamment par Holmer, Marzuola et Zworski [16, 17].

L'équation stationnaire correspondant à (5) est

$$-\partial_{xx} u + \omega u - \gamma u\delta - |u|^{p-1}u = 0.$$

Pour $\omega > \gamma^2/4$, cette équation admet une solution positive, explicite, unique, donnée par (voir [10, 11, 13])

$$\varphi_\omega(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}.$$

Puisque φ_ω est connue explicitement, le calcul de la dérivée du carré de la norme $L^2(\mathbb{R}^N)$ de φ_ω en fonction de ω est possible et la méthode de Grillakis, Shatah et Strauss s'avère naturellement la plus adaptée pour l'étude de la stabilité ou de l'instabilité des ondes stationnaires de (5). Néanmoins, dans la résolution de ce problème, un obstacle majeur demeure : déterminer le critère spectral, c'est à dire de déterminer le nombre de valeurs propres négatives de $S''(\varphi_\omega)$, où

$$S(v) = \frac{1}{2}\|\partial_x v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}.$$

Le travail présenté dans le chapitre 2 est motivé par les questions laissées ouvertes dans l'étude récente de Fukuizumi et Jeanjean [10]. En particulier, dans [10], les auteurs établissent une caractérisation variationnelle de φ_ω comme minimiseur de $S(v)$ sur une certaine contrainte et s'en servent pour déterminer le critère spectral. Dans le cas $\gamma > 0$, leur méthode permet de retrouver de manière simple les résultats déjà obtenus par Fukuizumi, Ohta et Ozawa [11]. Il n'en va pas de même dans le cas $\gamma < 0$ où ils sont contraints de considérer la stabilité uniquement pour des perturbations radiales.

Dans le chapitre 2, nous abordons l'étude du critère spectral sous un autre angle. En s'appuyant sur le fait que le spectre de $S''(\varphi_\omega)$ est connu depuis les travaux de Weinstein [35] quand $\gamma = 0$, on analyse son comportement lorsqu'on perturbe légèrement γ en positif ou en négatif. Une partie centrale du travail consiste à prouver que le spectre de l'opérateur $S''(\varphi_\omega)$ varie en fonction de γ de façon suffisamment régulière pour pouvoir faire cette analyse. Ensuite, on étend ce résultat à tous les paramètres γ en utilisant le fait que le noyau de l'opérateur $S''(\varphi_\omega)$ est réduit à $\{0\}$ lorsque $\gamma \neq 0$ et agit comme une barrière pour les valeurs propres. Combinée avec le calcul de la dérivée du carré de la norme $L^2(\mathbb{R}^N)$ de φ_ω, cette analyse spectrale permet de retrouver les résultats de [10, 11] et d'obtenir un tableau complet de la stabilité ou de l'instabilité de l'onde stationnaire en fonction des différentes valeurs des paramètres ω et γ. En particulier, dans les cas qui étaient restés ouverts jusqu'à présent, on obtient

Théorème 3. *Soit $\gamma < 0$. Il existe $\omega_2 > \gamma^2/4$ tel que l'onde stationnaire $e^{i\omega t}\varphi_\omega(x)$ est instable dans $H^1(\mathbb{R})$ pour tout $\omega > \gamma^2/4$ si $1 < p \leqslant 3$ et pour tout $\omega > \omega_2$ si $3 < p < 5$.*

Il est naturel de vouloir en savoir plus sur la nature de l'instabilité mise en évidence dans le Théorème 3 et les travaux [10, 11]. Néanmoins, l'un des inconvénients de la théorie de Grillakis, Shatah et Strauss est qu'elle donne très peu d'éléments de réponse à la question : comment se manifeste l'instabilité des ondes stationnaires ? Une première étape pour répondre à cette question consiste à rechercher les cas où l'onde stationnaire est instable par explosion. Précisément, on cherche à construire une suite de données initiales (u_n) convergeant vers φ_ω dans $H^1(\mathbb{R})$ et telle que la norme $H^1(\mathbb{R})$ de la solution de (5) avec pour donnée initiale u_n explose en temps fini. Notre résultat est le suivant.

Théorème 4. *Soit $\gamma \leqslant 0$, $\omega > \gamma^2/4$ et $p \geqslant 5$. Alors l'onde stationnaire $e^{i\omega t}\varphi_\omega(x)$ solution de (5) est instable par explosion.*

Comme beaucoup de résultats mettant en évidence un phénomène d'explosion, la preuve du Théorème 4 fait intervenir un résultat de type *identité du viriel* :

$$\partial_{tt}\|xu(t)\|_2^2 = 8Q(u(t)) \tag{6}$$

où $Q(v) = \|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}$ pour $v \in H^1(\mathbb{R})$. Pour justifier les calculs formels conduisant à (6), la plupart des preuves font intervenir la régularité $H^2(\mathbb{R})$ du problème d'évolution. Cependant, dans le cas de (5), cette régularité fait défaut en raison de la présence de la masse de Dirac. Pour contourner cette difficulté, nous prouvons (6) par une méthode d'approximation de la masse de Dirac par des potentiels plus réguliers pour lesquels le résultat de viriel est connu.

Pour la preuve du Théorème 4, on se base sur la méthode introduite en 1981 par Berestycki et Cazenave [2]. Il s'agit de définir un ensemble de données initiales générant chacune une solution explosive de (5) et de montrer qu'on peut prendre ces données aussi proches de φ_ω que désiré. Au cœur de la preuve de [2] est le fait que l'état fondamental est un minimiseur de S sur la contrainte $\{Q(v) = 0\}$. Dans notre cas, il est possible, mais long et délicat, de montrer que c'est encore vrai lorsque $5 < p < +\infty$, mais le cas $p = 5$ semble hors de portée. Alternativement, notre méthode, qui consiste à introduire une seconde contrainte, permet de contourner aisément cette difficulté.

Le Théorème 4 donne une caractérisation du phénomène d'instabilité lorsque $p \geqslant 5$. Néanmoins, lorsque $1 < p < 5$, il n'est pas difficile de montrer en utilisant l'inégalité de Gagliardo-Nirenberg et les lois de conservation que les solutions sont globales. En particulier, cela interdit tout phénomène d'instabilité par explosion. Pour compléter l'étude analytique de (5), des simulations numériques réalisées par Gadi Fibich et son équipe sont présentées à la fin du chapitre 2. Les résultats qu'ils ont obtenus montrent notamment que l'instabilité du Théorème 4 peut se manifester de deux manières différentes, éventuellement combinées : par dérive de la solution en s'éloignant de la masse de Dirac, ou bien par un début d'explosion suivi d'une forme d'oscillation autour d'un état stable.

En analysant la preuve du Théorème 4, on s'aperçoit que la méthode employée n'est pas liée à la dimension 1 et simplifie pour une non-linéarité de type puissance $f(x,s) = |s|^{p-1}s$ la preuve classique de [2] détaillée par Cazenave dans [6, section 8.2]. Or, dans [2], les auteurs ne se restreignent pas au cas des puissances et considèrent une large classe de non-linéarités. Il s'avère que l'approche de la preuve du Théorème 4 peut également s'étendre à des situations où la non-linéarité est générale

$$iu_t + \Delta u + f(u) = 0 \tag{7}$$

avec l'équation stationnaire correspondante

$$-\Delta\varphi + \omega\varphi = f(\varphi). \tag{8}$$

On retrouve alors de façon plus simple le résultat de [2] en simplifiant légèrement ses hypothèses. C'est le résultat principal du chapitre 3.

Théorème 5. *On suppose que f vérifie certaines hypothèses, notamment que la fonction $h(s) := (sf(s) - 2F(s))s^{-(2+4/N)}$ est strictement croissante sur $[0, +\infty)$ et $\lim_{s\to 0} h(s) = 0$.*

Alors pour tout état fondamental φ_ω de (8), l'onde stationnaire $e^{i\omega t}\varphi_\omega(x)$ solution de (7) est instable par explosion.

Outre l'introduction d'une double contrainte, l'un des ingrédients principaux de notre preuve est l'utilisation des résultats de Jeanjean et Tanaka [20, 21] en 2003. Ces résultats disent que, pratiquement sous les hypothèses de [3, 4], la fonctionnelle

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \int_{\mathbb{R}^N} F(v)dx$$

a une *géométrie de col*, c'est à dire que

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0,\ S(\gamma(1)) < 0\} \neq \emptyset, \tag{9}$$

$$\text{et} \quad c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} S(\gamma(t)) > 0.$$

De plus, on a l'identité

$$m = c$$

entre le niveau de moindre énergie m et le *niveau de col* c.

La particularité essentielle de notre preuve est que nous ne résolvons jamais explicitement de problème de minimisation. Nous utilisons juste les résultats de Jeanjean et Tanaka pour faire le lien entre les différents problèmes de minimisation que nous sommes amenés à considérer.

En collaboration avec Louis Jeanjean, nous avons cherché à savoir si les travaux [20, 21] ne pouvaient pas être exploités dans d'autres contextes, c'est l'objet du quatrième chapitre de cette thèse. Ce chapitre est consacré à l'étude de questions d'instabilité pour l'équation de Klein-Gordon. Néanmoins, l'idée générale qui traverse ce chapitre est que l'emploi de méthodes variationnelles récentes peut se révéler fructueux dans les études de stabilité ou d'instabilité pour les équations de Klein-Gordon ou Schrödinger comme pour d'autres équations « à ondes solitaires ».

Nous illustrons l'utilisation des résultats de [20, 21] dans deux situations. Dans la première, motivés par des travaux récents sur l'équation de Klein-Gordon [25, 26, 27], nous établissons une caractérisation variationnelle des états fondamentaux comme minimiseurs de S sur une grande famille de contraintes. L'équation d'évolution considérée est l'équation de Klein-Gordon non linéaire avec une non-linéarité de type puissance

$$u_{tt} - \Delta u + u = |u|^{p-1}u$$

et l'équation stationnaire correspondante, pour $\omega^2 < 1$, est

$$-\Delta\varphi_\omega + (1-\omega^2)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0. \tag{10}$$

Les travaux [25, 26, 27] présentent différents résultats d'instabilité par explosion en temps fini ou infini. Chacune des preuves fait intervenir une ou plusieurs caractérisations variationnelles des états fondamentaux de (10) comme minimiseurs de

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1-\omega^2}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}$$

sur certaines contraintes $\mathscr{K}_{\alpha,\beta}$. La définition de ces contraintes est semblable à chaque fois : pour un couple de réel (α,β), on pose

$$\mathscr{K}_{\alpha,\beta} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} \big| K_{\alpha,\beta}(v) = 0\}$$

où $K_{\alpha,\beta}(v) := \frac{\partial}{\partial\lambda} S(\lambda^\alpha v(\lambda^\beta \cdot))_{|\lambda=1} = 0$.

Bien qu'elles suivent des schémas similaires, les preuves des résultats de minimisation dans [25, 26, 27] soulèvent chacune des difficultés différentes, en particulier pour l'élimination du paramètre de Lagrange. Au contraire, notre méthode donne une preuve unifiée et courte pour une grande gamme de paramètres (α,β).

Théorème 6. *Soit $\alpha,\beta \in \mathbb{R}$ tels que*

$$\begin{cases} \quad \beta < 0, & \alpha(p-1) - 2\beta \geqslant 0 \text{ et } 2\alpha - \beta(N-2) > 0 \\ \text{ou} \ \ \beta \geqslant 0, & \alpha(p-1) - 2\beta \geqslant 0 \text{ et } 2\alpha - \beta N > 0. \end{cases}$$

Soit $\omega \in (-1,1)$ et φ_ω un état fondamental de (10). Alors

$$S(\varphi_\omega) = \min\{S(v) \big| v \in \mathscr{K}_{\alpha,\beta}\}.$$

L'idée de la preuve est la suivante : pour chaque $v \in \mathscr{K}_{\alpha,\beta}$, on construit un chemin $\gamma \in \Gamma$ (voir (9) pour la définition de Γ) tel que S atteint son maximum sur γ en v. Cela permet d'en déduire que

$$c \leqslant \min\{S(v) \big| v \in \mathscr{K}_{\alpha,\beta}\}.$$

On conclut en utilisant le fait que $c = m = S(\varphi_\omega)$ et $\varphi_\omega \in \mathscr{K}_{\alpha,\beta}$.

Pour notre deuxième illustration, on considère une équation de Klein-Gordon avec une non-linéarité générale

$$u_{tt} - \Delta u = g(u). \tag{11}$$

En 1985, Shatah [30] a montré en dimension $N \geqslant 3$ l'instabilité par explosion des solutions stationnaires de (11) qui sont aussi des états fondamentaux de

$$-\Delta\varphi = g(\varphi). \tag{12}$$

Les hypothèses sur g sont quasiment celles nécessaires pour assurer l'existence d'un état fondamental de (12). Nous montrons que le même type de résultat est également valable lorsque $N = 2$.

Théorème 7. *Pour $N = 2$, on suppose que g vérifie certaines hypothèses, notamment celles garantissant l'existence d'un état fondamental φ de (12). Alors φ vu comme une solution stationnaire de (11) est instable par explosion.*

L'une des différences principales entre le cas $N = 2$ et le cas $N \geqslant 3$ est liée à l'identité de Pohozaev : toute solution v de (12) vérifie

$$\frac{N-2}{2}\|\nabla v\|_2^2 = N \int_{\mathbb{R}^N} G(v)dx.$$

Lorsque $N = 2$, le membre de droite s'annule et on ne peut plus contrôler $\|\nabla v\|_2^2$. Notre preuve permet de surmonter cette difficulté.

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Chapitre 1

An existence and stability result for standing waves of nonlinear Schrödinger equations

Abstract. We consider a nonlinear Schrödinger equation with a nonlinearity of the form $V(x)g(u)$. Assuming that $V(x)$ behaves like $|x|^{-b}$ at infinity and $g(s)$ like $|s|^{p-1}s$ around 0, we prove the existence and orbital stability of travelling waves if $1 < p < 1 + (4-2b)/N$.

AMS Subject Classifications : 35J60, 35Q55, 37K45, 35B32

1.1 Introduction

This paper concerns the existence and orbital stability of standing waves for the nonlinear Schrödinger equation

$$iu_t + \Delta u + V(x)g(u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \ N \geqslant 3. \tag{1.1}$$

Here $u(t) \in H^1(\mathbb{R}^N, \mathbb{C})$, V is a real-valued potential and g is a nonlinearity satisfying $g(e^{i\theta}s) = e^{i\theta}g(s)$ for $s \in \mathbb{R}$.

A solution of the form $u(t,x) = e^{i\lambda t}\varphi(x)$ where $\lambda \in \mathbb{R}$ is called a standing wave. For solutions of this type with $\varphi \in H^1(\mathbb{R}^N, \mathbb{R})$, (1.1) is equivalent to

$$-\Delta\varphi + \lambda\varphi = V(x)g(\varphi), \quad \varphi \in H^1(\mathbb{R}^N, \mathbb{R}). \tag{1.2}$$

We are interested in the existence of positive solutions for (1.2) for small $\lambda > 0$. In addition we study the stability of the corresponding solutions of (1.1).

In the autonomous case, i.e. when V is a constant, we refer to the fundamental paper of Berestycki and Lions [2] where sufficient and almost necessary conditions are derived for the existence in $H^1(\mathbb{R}^N, \mathbb{R})$ of a solution of (1.2). When (1.2) is non autonomous, only partial results are known. A major difficulty to overcome is the lack of a priori bounds for the solutions. In contrast to the autonomous case where using dilations and taking advantage of Pohozaev identity is at the heart of the results of [2], no such device is available when V is non constant. Accordingly, most of the works dealing with existence require g to be of power type, i.e. $g(\varphi) = |\varphi|^{p-1}\varphi$ for a $p > 1$, or to satisfy the so-called Ambrosetti-Rabinowitz superquadraticity condition :

$$\exists \mu > 2 \text{ such that } 0 \leqslant \mu G(s) \leqslant g(s)s, \forall s \geqslant 0, \text{ where } G(s) = \int_0^s g(t)dt.$$

In this paper we prove the existence of solutions of (1.2), for small $\lambda > 0$, under the following assumptions (H1)-(H4) where $0 < b < 2$ and $1 < p < 1 + \frac{4-2b}{N-2}$,

(H1) there exists $\gamma > 2N/\{(N+2) - (N-2)p\}$ such that $V \in L^{\gamma}_{loc}(\mathbb{R}^N)$;

(H2) $\lim\limits_{|x| \to +\infty} V(x)|x|^b = 1$;

(H3) there exists $\varepsilon > 0$ such that $g : [0, \varepsilon] \to \mathbb{R}$ is continuous;

(H4) $\lim\limits_{s \to 0^+} \dfrac{g(s)}{s^p} = 1.$

Our approach is variational. Since only conditions around 0 are imposed on g, a first step will be to suitably extend g on all $\mathbb{R}$. This leads to study a modified problem but, as we shall see, the solutions we obtain for the modified problem have the property to converge to zero in the $L^\infty(\mathbb{R}^N)-$norm as λ decrease to zero. Thus, for sufficiently small $\lambda > 0$, they correspond to solutions of (1.2).

To get a solution of the modified equation we still face a lack of a priori bounds. To overcome this difficulty we borrow and further develop a method introduced by Berti and Bolle in a paper [3] which studies nonlinear wave equations. This method, roughly, make it possible to show the boundedness of Palais-Smale sequences at the mountain pass level for a class of functionals having a geometry *sufficiently* close to the one of the functional corresponding to the case $g(\varphi) = |\varphi|^{p-1}\varphi$. It relies on penalizing the functional outside the region where one expects to find a critical point. Our existence result is the following.

Theorem 1.1. *Assume (H1)-(H4). Then, there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0]$, *(1.2) has a non-trivial solution* φ_λ. *Furthermore,* φ_λ *has the following properties.*

1. *For all* $x \in \mathbb{R}^N$, $\varphi_\lambda \geqslant 0$.

2. *When* $\lambda \to 0$, $||\nabla \varphi_\lambda||_{L^2(\mathbb{R}^N)} \to 0$ *and* $||\varphi_\lambda||_{L^\infty(\mathbb{R}^N)} \to 0$.

Since our solutions converge to zero in $\dot{H}^1(\mathbb{R}^N, \mathbb{R})$ and $L^\infty(\mathbb{R}^N)$ as $\lambda \to 0$, 0 is a bifurcation point of (1.2). With our approach we can (see Remark 1.9) obtain sharp estimates on the $L^p(\mathbb{R}^N)-$bifurcation of our solutions as $\lambda \to 0$. We refer to [14, 21] for previous bifurcations results.

Once the existence of solutions of (1.2) is proved we consider the stability of the associated travelling waves. The study of the orbital stability of solutions of (1.1) has seen the contributions of many authors. It is of particular significance for physical reasons and we refer the reader to the introductions of [9, 20, 22] for motivations of studying this problem. In the case V constant and $g(u) = |u|^{p-1}u$, Cazenave and Lions [5] proved the stability of the ground state solutions of (1.2) when $1 < p < 1 + \frac{4}{N}$ and for any $\lambda > 0$. On the contrary, when $1 + \frac{4}{N} < p < 1 + \frac{4}{N-2}$, Berestycki and Cazenave [1] showed the instability of bounded states of (1.2) and when $p = 1 + \frac{4}{N}$, Weinstein [24] proved that instability also holds. We also mention [12] for a general stability theory for solitary waves of Hamiltonian systems.

In [5] both the autonomous character of (1.2) and the fact that g is homogeneous are essential in the proofs. Also dealing with an homogeneous and to some extend autonomous nonlinearity seems essential to use directly the results of [12] (see nevertheless [18]). When (1.2) is non autonomous only partial results are known so far (see [4, 8, 9, 13, 20, 22] and the references therein). Directly related to our stability result is a recent work of de Bouard and Fukuizumi [6] where stability of positive ground states of (1.2) is obtain for $g(u) = |u|^{p-1}u$ under the following conditions on V :

(V1) $V \geqslant 0$, $V \not\equiv 0$, $V \in \mathcal{C}(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$, $V \in L^{\theta^*}(|x| \leqslant 1)$, where $\theta^* = 2N/\{(N+2) - (N-2)p\}$,

(V2) There exists $b \in (0, 2)$, $C > 0$ and $a > \{(N+2) - (N-2)p\}/2 > b$ such that $\left|\left(V(x) - |x|^{-b}\right)\right| \leqslant C|x|^{-a}$ for all x with $|x| \geqslant 1$.

Under these assumptions and if $1 < p < 1 + (4-2b)/(N-2)$ the existence of ground states solutions follows immediately from the existing literature. In [6] de Bouard and Fukuizumi proved that the corresponding standing waves are stable if $1 < p < 1 + (4-2b)/N$ and $\lambda > 0$ is small.

Our stability result, Theorem 1.2, extends the result of [6]. If we do borrow some arguments from this paper, new ingredients are necessary to derive Theorem 1.2. In particular, the fact that we do not know if the solutions obtained in Theorem 1.1 are ground states is a new major difficulty. To state our stability result we need some definitions and preliminary results. First, to check that the local Cauchy problem is well posed for (1.1), in addition to (H1)-(H4), we require on g

(H5) $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$;

(H6) there exist $C > 0$ and $\alpha \in [0, \frac{4}{N-2})$ such that $\limsup_{|s| \to +\infty} \dfrac{|g'(s)|}{|s|^\alpha} \leqslant C$.

Clearly (H5)-(H6) are sufficient to guarantee that the condition

$$|g(v) - g(u)| \leqslant C(1 + |v|^\alpha + |u|^\alpha)|v - u| \text{ for all } u, v \in \mathbb{R}$$

introduced in Remark 4.3.2 of [4] holds. By [4] we then know that the Cauchy problem for (1.1) is locally well posed.

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$ we write $v = v_1 + iv_2$. The space $H^1(\mathbb{R}^N, \mathbb{C})$ will be equipped with the norm

$$||v|| = \sqrt{||v||_2^2 + ||\nabla v||_2^2}$$

where $||v||_2^2 = |v_1|_2^2 + |v_2|_2^2$ and $||\nabla v||_2^2 = |\nabla v_1|_2^2 + |\nabla v_2|_2^2$. Here and elsewhere $|\cdot|_p$ denotes the usual norm on $L^p(\mathbb{R}^N, \mathbb{R})$. We also define on $L^2(\mathbb{R}^N, \mathbb{C})$ the scalar product

$$\langle u, v \rangle_2 = \int_{\mathbb{R}^N} \mathrm{Re}(u(x)\overline{v(x)})dx.$$

Finally, let the energy functional E and the charge Q on $H^1(\mathbb{R}^N, \mathbb{C})$ be given by

$$E(v) = \frac{1}{2}||\nabla v||_2^2 - \int_{\mathbb{R}^N} V(x)G(v)dx \text{ and } Q(v) = \frac{1}{2}||v||_2^2$$

where $G(z) = \displaystyle\int_0^{|z|} g(t)dt$ for all $z \in \mathbb{C}$. It follows from [4] that

Proposition 1.1. *Assume (H1)-(H6). Then, for every* $u_0 \in H^1(\mathbb{R}^N, \mathbb{C})$ *there exist* $T_{u_0} > 0$ *and a unique solution* $u(t) \in \mathcal{C}([0, T_{u_0}), H^1(\mathbb{R}^N, \mathbb{C}))$ *with* $u(0) = u_0$ *satisfying*

$$E(u(t)) = E(u_0), \qquad Q(u(t)) = Q(u_0), \qquad \textit{for all } t \in [0, T_{u_0}).$$

Finally we require a stronger version of (H4).

(H7) $\displaystyle\lim_{s \to 0^+} \frac{g'(s)}{ps^{p-1}} = 1.$

Now by stability we mean

Definition 1.2. *Let* φ_λ *be a solution of (1.2). We say that the travelling wave* $u(x,t) = e^{i\lambda t}\varphi_\lambda(x)$ *associated to* φ_λ *is* stable *in* $H^1(\mathbb{R}^N, \mathbb{C})$ *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the following property. If* $u_0 \in H^1(\mathbb{R}^N, \mathbb{C})$ *is such that* $||u_0 - \varphi_\lambda|| < \delta$ *and* $u(t)$ *is a solution of (1.1) in some interval* $[0, T_{u_0})$ *with* $u(0) = u_0$*, then* $u(t)$ *can be continued to a solution in* $[0, +\infty)$ *and*

$$\sup_{t \in [0, +\infty)} \inf_{\theta \in \mathbb{R}} ||u(t) - e^{i\theta}\varphi_\lambda|| < \varepsilon.$$

Our result is the following

Theorem 1.2. *Assume (H1)-(H7),* $1 < p < 1 + \frac{4-2b}{N}$*, and let* (φ_λ) *be the family of solutions of (1.2) obtained in Theorem 1.1. Then there exists* $\tilde{\lambda} > 0$ *such that for all* $\lambda \in (0, \tilde{\lambda}]$ *the travelling wave* $e^{i\lambda t}\varphi_\lambda(x)$ *is stable in* $H^1(\mathbb{R}^N, \mathbb{C})$*.*

From Theorem 1.2 we see that, for $\lambda > 0$ small enough, stability only depends on the behaviour of V at infinity and of g around zero. Indeed, as it is shown in [10], when $V(x) = |x|^{-b}$ instability occurs for $g(u) = |u|^{p-1}u$ if $p > 1 + \frac{4-2b}{N}$. To our knowledge, Theorem 1.2 is the first result to enlighten this fact.

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$ and $\lambda > 0$ let

$$S_\lambda(v) = \frac{1}{2}(||\nabla v||_2^2 + \lambda ||v||_2^2) - \int_{\mathbb{R}^N} V(x)G(v)dx.$$

Under our assumptions it is standard to check that S_λ is $\mathcal{C}^2$. Our proof of Theorem 1.2 relies on the following stability criterion established in [12].

Proposition 1.3. *Assume (H1)-(H7) and let φ_λ be a solution of (1.2). If there exists $\delta > 0$ such that for every $v \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfying $\langle \varphi_\lambda, v \rangle_2 = 0$ and $\langle i\varphi_\lambda, v \rangle_2 = 0$ we have*

$$\langle S''_\lambda(\varphi_\lambda)v, v \rangle \geqslant \delta ||v||^2,$$

then the standing wave $e^{i\lambda t}\varphi_\lambda(x)$ is stable in $H^1(\mathbb{R}^N, \mathbb{C})$.

To check this criterion, following an approach laid down in [7], we first show, in Subsection 1.3.1, that our solutions (φ_λ) properly rescaled converge in $H^1(\mathbb{R}^N)$ to the unique positive solution $\psi \in H^1(\mathbb{R}^N, \mathbb{R})$ of the limit equation

$$-\Delta u + u = \frac{1}{|x|^b}|u|^{p-1}u, \;\; u \in H^1(\mathbb{R}^N, \mathbb{R}). \tag{1.3}$$

Then we derive, see Subsection 1.3.2, some properties of $\psi \in H^1(\mathbb{R}^N, \mathbb{R})$, in particular we show that it is non-degenerate. Finally, in Subsection 3.3, we show that the conclusion of Proposition 1.3 holds.

The paper is organized as follows. In Section 1.2 we establish Theorem 1.1 and in Section 1.3 we prove Theorem 1.2. An uniqueness result which is necessary for the proof of Theorem 1.2 is establish, using results of [26], in the Appendix.

Notations Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also we make the convention that when we take a subsequence of a sequence (u_n) we denote it again by (u_n).

1.2 Existence

This section is devoted to the proof of Theorem 1.1. For this we use a variational approach and consequently a first step is to extend the nonlinearity g outside of $[0, \varepsilon]$. Let $H \equiv H^1(\mathbb{R}^N, \mathbb{R})$ be equipped with its standard norm $|\cdot|_H$. We consider the modified problem

$$-\Delta v + \lambda v = V(x)f(v), \; v \in H \tag{1.4}$$

where

$$f(s) = \begin{cases} g(\varepsilon) \;\text{ if } s \geqslant \varepsilon \\ g(s) \;\text{ if } s \in [0, \varepsilon] \\ 0 \;\text{ if } s \leqslant 0. \end{cases}$$

It is convenient to write (1.4) as

$$-\Delta v+\lambda v=V(x)\left(v_+^p+r(v)\right),\ v\in H \tag{1.5}$$

with $v_+=\max\{v,0\}$ and $r(s)=f(s)-s_+^p$.

To develop our variational procedure we rescaled (1.5) in order to eliminate $\lambda>0$ from the linear part. For $v\in H$, let $\tilde v\in H$ be such that

$$v(x)=\lambda^{\frac{2-b}{2(p-1)}}\tilde v(\sqrt{\lambda}x). \tag{1.6}$$

Clearly $v\in H$ satisfies (1.5) if and only if $\tilde v\in H$ satisfies

$$-\Delta\tilde v+\tilde v=V_\lambda(x)\tilde v_+^p+V(\frac{x}{\sqrt{\lambda}})\tilde r(\tilde v) \tag{1.7}$$

where

$$\tilde r(s)=\lambda^{-\frac{2-b}{2(p-1)}-1}r(\lambda^{\frac{2-b}{2(p-1)}}s)\ \text{ and }\ V_\lambda(x)=\lambda^{-b/2}V(x/\sqrt{\lambda}). \tag{1.8}$$

A solution of (1.7) will be obtained as a critical point of the functional $\tilde S_\lambda: H\to\mathbb{R}$ given by

$$\tilde S_\lambda(v)=\frac{1}{2}|v|_H^2-\frac{1}{p+1}\int_{\mathbb{R}^N}V_\lambda(x)v(x)_+^{p+1}dx-\tilde R_\lambda(v)$$

with $\tilde R_\lambda(v)=\displaystyle\int_{\mathbb{R}^N}\lambda^{b/2}V_\lambda(x)\left(\int_0^{|v|}\tilde r(t)dt\right)dx.$

By (H1) we can fix a $p'\in(p,1+(4-2b)/(N-2))$ such that $2N/\{(N+2)-(N-2)p'\}<\gamma$. The following estimate will be crucial throughout the paper.

Lemma 1.4. *Assume (H1)-(H4). Then for any $q\in[1,p']$ there exists $C>0$ such that for any $\lambda>0$ sufficiently small and all $v\in H$,*

$$\left|\int_{\mathbb{R}^N}V_\lambda(x)|v(x)|^{q+1}dx\right|\leqslant C|v|_H^{q+1}.$$

Proof. By the assumptions (H1)-(H2) there exists $R>0$ such that

$$|V(x)|\leqslant 2|x|^{-b},\ \forall\,|x|\geqslant R\ \text{ and }\ V\in L^\gamma(B(R)). \tag{1.9}$$

Here $B(R)=\{x\in\mathbb{R}^N:|x|<R\}$. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}V_\lambda(x)|v(x)|^{q+1}dx\right| &\leqslant \left|\int_{B(R)}V_\lambda(x)|v(x)|^{q+1}dx\right|\\ &\quad+\left|\int_{\mathbb{R}^N\setminus B(R)}V_\lambda(x)|v(x)|^{q+1}dx\right|.\end{aligned} \tag{1.10}$$

By Hölder's inequality,

$$\left|\int_{B(R)} V_\lambda(x)|v(x)|^{q+1}dx\right| \leqslant |V_\lambda|_{L^\theta(B(R))}\,|v|_{2^*}^{q+1} \tag{1.11}$$

with $\theta = 2N/\{(N+2)-(N-2)q\}$. But

$$|V_\lambda|^\theta_{L^\theta(B(R))} = |V_\lambda|^\theta_{L^\theta(B(\sqrt{\lambda}R))} + |V_\lambda|^\theta_{L^\theta(B(R)\backslash B(\sqrt{\lambda}R))} \tag{1.12}$$

and, since $|V_\lambda|^\theta_{L^\theta(B(\sqrt{\lambda}R))} = \lambda^{-b\theta/2+N/2}\,|V|_{L^\theta(B(R))}$ with $-b\theta/2+N/2>0$, we can assume that

$$|V_\lambda|_{L^\theta(B(\sqrt{\lambda}R))} \leqslant 1. \tag{1.13}$$

Also, from (1.9) it follows that $V_\lambda(x) \leqslant 2|x|^{-b}$ on $\mathbb{R}^N\backslash B(\sqrt{\lambda}R)$. Thus

$$|V_\lambda|_{L^\theta(B(R)\backslash B(\sqrt{\lambda}R))} \leqslant |\frac{2}{|x|^b}|_{L^\theta(B(R))} \leqslant C, \tag{1.14}$$

and

$$\left|\int_{\mathbb{R}^N\backslash B(R)} V_\lambda(x)|v(x)|^{q+1}dx\right| \leqslant C|v|^{q+1}_{q+1}. \tag{1.15}$$

Now, combining (1.10)-(1.15) and using Sobolev's embeddings we get the required estimate. □

A first consequence of Lemma 1.4 is the following estimate on the "rest" $\tilde{R}_\lambda$ of the functional $\tilde{S}_\lambda$.

Lemma 1.5. *Assume (H1)-(H4). Then there exist $C>0$ and $\alpha>0$ such that for all $a>0$ there exists $A>0$ such that*

$$|\tilde{R}_\lambda(v)| + |\nabla\tilde{R}_\lambda(v)v| \leqslant C(a|v|^{p+1}_H + \lambda^\alpha A|v|^{p'+1}_H) \tag{1.16}$$

for all $\lambda>0$ sufficiently small and all $v \in H$.

Proof. From the definition of r and (H4), we see that for any $a>0$ there exists $A>0$ such that

$$|r(s)| \leqslant a|s|^p + A|s|^{p'},\ \forall s\in\mathbb{R}. \tag{1.17}$$

This implies, see (1.8), that

$$|\tilde{r}(s)| \leqslant \lambda^{-b/2}a|s|^p + \lambda^{-b/2}\lambda^\alpha A|s|^{p'},\ \forall s\in\mathbb{R} \tag{1.18}$$

with $\alpha = \dfrac{(p'-p)(2-b)}{2(p-1)} > 0$. As a consequence, for any $v\in H$,

$$|\tilde{R}_\lambda(v)| \leqslant \frac{a}{p+1}\int_{\mathbb{R}^N}|V_\lambda(x)||v(x)|^{p+1}dx + \frac{\lambda^\alpha A}{p'+1}\int_{\mathbb{R}^N}|V_\lambda(x)||v(x)|^{p'+1}dx$$

and using Lemma 1.4 we get that

$$|\tilde{R}_\lambda(v)| \leqslant C(a|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}). \tag{1.19}$$

Analogously, we can prove that

$$|\nabla\tilde{R}_\lambda(v)v| \leqslant C(a|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}). \tag{1.20}$$

Combining (1.19) and (1.20) finishes the proof. □

We shall obtain a critical point of $\tilde{S}_\lambda$ by a mountain pass type argument. However, even though it is likely that $\tilde{S}_\lambda$ has a mountain pass geometry, showing that the Palais-Smale sequences at the mountain pass level are bounded seems out of reach under our weak assumptions on g. To overcome this difficulty we develop an approach, inspired by [3], which consists in truncating the remainder term of $\tilde{S}_\lambda$ outside of a ball centered at the origin and to show that, as $\lambda > 0$ goes to zero, all Palais-Smale sequences at the mountain-pass level lie in this ball. Precisely, let $T > 0$ be the truncation radius (its value will be indicated later) and consider a smooth function $\nu : [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} \nu(s) = 1 & \text{for} \quad s \in [0,1], \\ 0 \leqslant \nu(s) \leqslant 1 & \text{for} \quad s \in [1,2], \\ \nu(s) = 0 & \text{for} \quad s \in [2,+\infty), \\ |\nu'|_\infty \leqslant 2. \end{cases}$$

For $v \in H$, we define

$$\widehat{S}_\lambda(v) = \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} V_\lambda(x) v(x)_+^{p+1} dx - \widehat{R}_\lambda(v),$$

where $\widehat{R}_\lambda(v) = t(v)\tilde{R}_\lambda(v)$ with $t(v) := \nu\left(\dfrac{|v|_H^2}{T^2}\right)$.

We have the following bounds on $\widehat{R}_\lambda(v)$ and $\nabla\widehat{R}_\lambda(v)v$

Lemma 1.6. *Assume (H1)-(H4). Then there exists $C > 0$ such that for all $a > 0$, there exists $A > 0$, satisfying for all $v \in H$*

$$\begin{aligned} |\widehat{R}_\lambda(v)| &\leqslant C(aT^{p+1} + \lambda^\alpha A T^{p'+1}), && (1.21)\\ |\nabla\widehat{R}_\lambda(v)v| &\leqslant C(aT^{p+1} + \lambda^\alpha A T^{p'+1}). && (1.22) \end{aligned}$$

Proof. Since $t(v) = 0$ for $|v|_H > \sqrt{2}T$, (1.21) follows directly from Lemma 1.5. Also $\nabla\widehat{R}_\lambda(v) = t(v)\nabla\tilde{R}_\lambda(v) + \tilde{R}_\lambda(v)\nabla t(v)$ with $\nabla t(v)v = 2\nu'(\dfrac{|v|_H^2}{T^2})\dfrac{|v|_H^2}{T^2}$ and thus we also have (1.22). □

Lemma 1.7. *Assume (H1)-(H4). Then there exists $\overline{\lambda} > 0$ such that for all $\lambda \in (0, \overline{\lambda}]$, $\widehat{S}_\lambda$ has a mountain pass geometry. Also $\widehat{S}_\lambda$ admits at the mountain pass level $c(\lambda) > 0$ a critical point $\tilde{\varphi}_\lambda \in H \setminus \{0\}$ which is also a critical point for $\tilde{S}_\lambda$. Moreover there exists $C > 0$ such that $|\tilde{\varphi}_\lambda|_H \leqslant C$, $\forall \lambda \in (0, \overline{\lambda}]$.*

Proof. Let us prove that $\widehat{S}_\lambda$ has a mountain pass geometry for any $\lambda > 0$ sufficiently small. Obviously, we have $\widehat{S}_\lambda(0) = 0$. Let $a > 0$. From Lemma 1.4 (used with $q = p$) and Lemma 1.5 there exists $A > 0$ such that for $v \in H$

$$\widehat{S}_\lambda(v) \geqslant \frac{1}{2}|v|_H^2 - C((1+a)|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}).$$

Thus, there exists $\delta > 0$ small and $m \geqslant 0$ such that $\widehat{S}_\lambda(v) > m > 0$ for all $v \in H$ satisfying $|v|_H = \delta$, uniformly in λ if λ is small enough.

Now let $\varpi \in \mathcal{C}_0^\infty(\mathbb{R}^N) \setminus \{0\}$ with $\varpi \geqslant 0$ and $\varpi = 0$ on $B(1)$. Because of (H2), there exists $R > 0$ such that

$$V(x) \geqslant \frac{1}{2|x|^b} \text{ if } |x| \geqslant R.$$

Thus, for $\lambda > 0$ small enough

$$\int_{\mathbb{R}^N} V_\lambda(x)\varpi(x)^{p+1}dx \geqslant \int_{\mathbb{R}^N} \frac{1}{2|x|^b}\varpi(x)^{p+1}dx.$$

Defining $\varpi_B := B\varpi$ we observe that for $B > 0$ large enough $\widehat{R}_\lambda(\varpi_B) = 0$. Thus letting $D = \frac{|\varpi|_H^2}{2}$ and $E = \int_{\mathbb{R}^N} \frac{1}{2|x|^b}\varpi(x)^{p+1}dx$ we have, for $B > 0$ large enough,

$$\widehat{S}_\lambda(\varpi_B) \leqslant DB^2 - EB^{p+1} < 0$$

for any $\lambda > 0$ sufficiently small.

Since $\widehat{S}_\lambda$ has a mountain pass geometry, defining

$$c(\lambda) := \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} \widehat{S}_\lambda(\gamma(s))$$

where $\Gamma := \{\gamma \in \mathcal{C}([0,1], H) \,|\, \gamma(0) = 0, \widehat{S}_\lambda(\gamma(1)) < 0\}$, Ekeland's principle gives the existence of a Palais-Smale sequence at the mountain pass level $c(\lambda)$. Namely of a sequence $(v_n) \subset H$ such that

$$\nabla \widehat{S}_\lambda(v_n) \to 0, \quad (1.23)$$
$$\widehat{S}_\lambda(v_n) \to c(\lambda). \quad (1.24)$$

Let us show that, if $\lambda > 0$ small enough, this Palais-Smale sequence lies, for $n \in \mathbb{N}$ large, in the ball of H where $\widehat{S}_\lambda$ and $\widetilde{S}_\lambda$ coincide. We begin by an estimate on the mountain pass level. For every $t \in [0,1]$ we have

$$\widehat{S}_\lambda(t\varpi_B) \leqslant DB^2t^2 - EB^{p+1}t^{p+1} + |\widehat{R}_\lambda(t\varpi_B)|.$$

Thanks to (1.21) and the definition of $c(\lambda)$ this gives

$$c(\lambda) \leqslant W + C(aT^{p+1} + A\lambda^\alpha T^{p'+1}) \tag{1.25}$$

with $W = D\left(\frac{2D}{(p+1)E}\right)^{\frac{2}{p-1}} - E\left(\frac{2D}{(p+1)E}\right)^{\frac{p+1}{p-1}}$. Note that the constants W and C are independent of $T > 0$ and of $\lambda > 0$ sufficiently small.

To prove that $\limsup_{n\to\infty} |v_n|_H < T$ we first show that (v_n) is bounded in H. Seeking a contradiction, we assume that, up to a subsequence, $|v_n|_H \to +\infty$. Therefore, for $n \in \mathbb{N}$ large enough, we have $|v_n|_H^2 > 2T^2$ and thus $\widehat{R}_\lambda(v_n) = \nabla\widehat{R}_\lambda(v_n)v_n = 0$. It follows that

$$2\widehat{S}_\lambda(v_n) - \nabla\widehat{S}_\lambda(v_n)v_n = \left(1 - \frac{2}{p+1}\right)\int_{\mathbb{R}^N} V_\lambda(x)(v_n(x))_+^{p+1}dx.$$

Furthermore, since $\widehat{S}_\lambda(v_n) \to c(\lambda)$, we can assume that $\widehat{S}_\lambda(v_n) \leqslant 2c(\lambda)$ and we get

$$\left(1 - \frac{2}{p+1}\right)\int_{\mathbb{R}^N} V_\lambda(x)(v_n(x))_+^{p+1}dx \leqslant 4c(\lambda) + \|\nabla\widehat{S}_\lambda(v_n)\||v_n|_H.$$

Consequently we have

$$\begin{aligned} |v_n|_H^2 &= \nabla\widehat{S}_\lambda(v_n)v_n + \int_{\mathbb{R}^N} V_\lambda(x)(v_n(x))_+^{p+1}dx \\ &\leqslant \left(1 + \frac{p+1}{p-1}\right)\|\nabla\widehat{S}_\lambda(v_n)\||v_n|_H + 4\left(\frac{p+1}{p-1}\right)c(\lambda) \end{aligned}$$

and therefore

$$|v_n|_H \leqslant \left(1 + \frac{p+1}{p-1}\right)\|\nabla\widehat{S}_\lambda(v_n)\| + 4\left(\frac{p+1}{p-1}\right)c(\lambda)|v_n|_H^{-1}.$$

Since the right member tends to 0 as $n \to \infty$ we have a contradiction. Thus (v_n) stays bounded in H and, in particular, $\nabla\widehat{S}_\lambda(v_n)v_n \to 0$.

Let us now show that $|v_n|_H < T$ for $n \in \mathbb{N}$ large. Note that, since $\hat{S}_\lambda$ (and thus (v_n)) depends on T, the value of T can not be changed. Still arguing by contradiction, we assume that $\lim_{n\to\infty} |v_n|_H \in [T, +\infty)$. We have

$$\widehat{S}_\lambda(v_n) - \frac{1}{p+1}\nabla\widehat{S}_\lambda(v_n)v_n = \left(\frac{1}{2} - \frac{1}{p+1}\right)|v_n|_H^2 - \widehat{R}_\lambda(v_n) + \frac{1}{p+1}\nabla\widehat{R}_\lambda(v_n)v_n. \tag{1.26}$$

Then using (1.21)-(1.25) and passing to the limit in (1.26), we obtain

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)T^2 \leqslant W + C(aT^{p+1}+A\lambda^\alpha T^{2^\star}).$$

At this point, choosing $a>0$ sufficiently small, we see that if $T^2 > \frac{2(p+1)}{p-1}W$ we obtain a contradiction when $\lambda>0$ is small enough. This proves that (v_n) lies in the region where $\tilde{S}_\lambda$ and $\widehat{S}_\lambda$ coincide.

Now since $(v_n)\subset H$ is bounded we can assume that $v_n \rightharpoonup v_\infty$ weakly in H. To end the proof we just need to show that $v_n \to v_\infty$ strongly in H. The condition $\nabla\widehat{S}_\lambda(v_n)\to 0$ is just

$$-\Delta v_n + v_n - V_\lambda(x)(v_n)_+^p - V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_n) \to 0 \text{ in } H^{-1}. \tag{1.27}$$

Because of the decrease of V to 0 at infinity we have, in a standard way, that

$$V_\lambda(x)(v_n)_+^p + V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_n) \to V_\lambda(x)(v_\infty)_+^p + V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_\infty) \text{ in } H^{-1}. \tag{1.28}$$

Now let $L : H \to H^{-1}$ be defined by

$$\langle Lu, v\rangle = \int_{\mathbb{R}^N}(\nabla u\nabla v + uv)dx.$$

The operator L is invertible, therefore, from (1.27)-(1.28),

$$v_n \to L^{-1}\left(V_\lambda(x)(v_\infty)_+^p + V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_\infty)\right).$$

By uniqueness of the limit, we have $v_n \to v_\infty$ in H and by continuity v_∞ is a solution of (1.7) at the mountain pass level $c(\lambda)$. We set $\tilde{\varphi}_\lambda = v_\infty$. At this point the lemma is proved. □

Lemma 1.8. *Assume (H1)-(H4). The solutions of (1.7), obtained in Lemma 1.7 have, in addition, the following properties*

(i) $|\tilde{\varphi}_\lambda|_\infty \leqslant C$, *for a* $C>0$ *independent of* $\lambda\in(0,\overline{\lambda}]$,

(ii) for all $x\in\mathbb{R}^N$, $\tilde{\varphi}_\lambda(x)\geqslant 0$.

Proof. Starting from (1.4) and the change of variables (1.6) we see that our solutions $\tilde{\varphi}_\lambda$ satisfy

$$-\Delta\tilde{\varphi}_\lambda + \tilde{\varphi}_\lambda = \lambda^{-\frac{2-b}{2(p-1)}-1}V(\frac{x}{\sqrt{\lambda}})f(\lambda^{\frac{2-b}{2(p-1)}}\tilde{\varphi}_\lambda). \tag{1.29}$$

We see from (H4) that $|f(s)| \leqslant C|s|^p$ for a $C>0$, $\forall s \in \mathbb{R}$. Thus

$$\left|\lambda^{-\frac{2-b}{2(p-1)}-1} V\Big(\frac{x}{\sqrt{\lambda}}\Big) f\big(\lambda^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_\lambda\big)\right| \leqslant C|V_\lambda(x)||\tilde{\varphi}_\lambda|^p \tag{1.30}$$

with a $C>0$, independent of $\lambda \in (0,\overline{\lambda}]$. To obtain (i) we follow a bootstrap argument. The crucial point is to insure that the estimates we get are independent of $\lambda \in (0,\overline{\lambda}]$.

Let $\theta = 2N/\{(N+2)-(N-2)p\}$. Assuming that $\tilde{\varphi}_\lambda \in L^q(\mathbb{R}^N)$ we claim that

(claim) $V_\lambda|\tilde{\varphi}_\lambda|^p \in L^r(\mathbb{R}^N)$ with $r = \frac{\theta q}{\theta p + q}$ and is bounded in $L^r(\mathbb{R}^N)$ as a function of $|\tilde{\varphi}_\lambda|_q$ only.

To see this we choose $R>0$ such that $|V(x)| \leqslant 2|x|^{-b}$, $\forall |x| \geqslant R$ and we write $\mathbb{R}^N = B(\sqrt{\lambda}R) \cup (B(R)\backslash B(\sqrt{\lambda}R)) \cup (\mathbb{R}^N \backslash B(R))$.

On $\mathbb{R}^N\backslash B(R)$ since $|V_\lambda(x)| \leqslant C$, for a $C>0$ we directly have

$$|V_\lambda||\tilde{\varphi}_\lambda|^p \in L^{\frac{q}{p}}(\mathbb{R}^N\backslash B(R))$$

and thus, since $V_\lambda \tilde{\varphi}_\lambda^p \in L^1(\mathbb{R}^N \setminus B(R))$ and $\dfrac{q}{p} > r$, we have by interpolation

$$|V_\lambda||\tilde{\varphi}_\lambda|^p \in L^r(\mathbb{R}^N\backslash B(R)).$$

On $B(R)\backslash B(\sqrt{\lambda}R)$ we have $|V_\lambda(x)| \leqslant 2|x|^{-b}$ with $|x|^{-b} \in L^\theta(B(R))$. Thus

$$\begin{aligned}\int_{B(R)\backslash B(\sqrt{\lambda}R)} |V_\lambda(x)|^r |\tilde{\varphi}_\lambda|^{rp} dx &\leqslant \left(\int_{B(R)} \frac{1}{|x|^{b\theta}} dx\right)^{\frac{q}{q+\theta p}} \left(\int_{B(R)} |\tilde{\varphi}_\lambda|^q dx\right)^{\frac{\theta p}{q+\theta p}} \\ &\leqslant C|\tilde{\varphi}_\lambda|_q^{\frac{\theta q p}{q+\theta p}}.\end{aligned}$$

On $B(\sqrt{\lambda}R)$ we have

$$\int_{B(\sqrt{\lambda}R)} |V_\lambda(x)|^r |\tilde{\varphi}_\lambda|^{rp} dx \leqslant \left(\int_{B(\sqrt{\lambda}R)} |V_\lambda(x)|^\theta dx\right)^{\frac{q}{q+\theta p}} \left(\int_{B(\sqrt{\lambda}R)} |\tilde{\varphi}_\lambda|^q dx\right)^{\frac{\theta p}{q+\theta p}}$$

with

$$|V_\lambda|^\theta_{L^\theta(B(\sqrt{\lambda}R))} = \lambda^{-b\theta/2+N/2}\, |V|^\theta_{L^\theta(B(R))} \to 0$$

and this proves our claim. Now since $V_\lambda|\tilde{\varphi}_\lambda|^p \in L^r(\mathbb{R}^N)$ we have $\tilde{\varphi}_\lambda \in W^{2,r}(\mathbb{R}^N)$ and thus $\tilde{\varphi}_\lambda \in L^t(\mathbb{R}^N)$ with $t = \frac{Nr}{N-2r}$.

It is now easy to check that, choosing $q = 2^*$, we have $t > q$ and that the bootstrap will give, in a finite number of steps, $r > \frac{N}{2}$ so that $\tilde{\varphi}_\lambda \in W^{2,r}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$. In addition, since for a $C > 0$, $|\tilde{\varphi}_\lambda|_H \leqslant C, \forall \lambda \in (0, \overline{\lambda}]$ we have, for a $C > 0$, $|\tilde{\varphi}_\lambda|_{2^*} \leqslant C, \forall \lambda \in (0, \overline{\lambda}]$ and by our claim the various constants of the Sobolev's embeddings are independent of $\lambda \in (0, \overline{\lambda}]$. This proves (i).

For (ii), we argue as follows. Let $\varphi = \varphi_+ - \varphi_-$ where $\varphi_+ = \max\{\varphi, 0\}$ and $\varphi_- = \max\{-\varphi, 0\}$ and suppose that φ satisfy

$$-\Delta\varphi + \varphi = V\left(\frac{x}{\sqrt{\lambda}}\right)\tilde{f}(\varphi)$$

with $\tilde{f} = 0$ if $s \leqslant 0$. We know that $\varphi_+, \varphi_- \in H$. Then, by multiplying by φ_- and integrating, we obtain

$$-\int_{\mathbb{R}^N} |\nabla\varphi_-|^2 - \varphi_-^2 = 0,$$

Therefore $\varphi_- = 0$. $\square$

Now we can give the

Proof of Theorem 1.1. Taking into account Lemmas 1.7 and 1.8 all that remains to show is that $|\varphi_\lambda|_H \to 0$ and $|\varphi_\lambda|_\infty \to 0$, as $\lambda \to 0$, when φ_λ is given by

$$\varphi_\lambda(x) = \lambda^{\frac{2-b}{2(p-1)}}\tilde{\varphi}_\lambda(\sqrt{\lambda}x).$$

Since $\frac{2-b}{2(p-1)} > 0$ we immediately get, from Lemma 1.8, that $|\varphi_\lambda|_\infty \to 0$ and this proves, in particular, that φ_λ is solution of (1.2) when $\lambda > 0$ is small enough. Now, since $p < 1 + \frac{4-2b}{N}$ we see from direct calculations that $|\varphi_\lambda|_H \to 0$. $\square$

Remark 1.9. We deduce from the proof of Theorem 1.1 that (1.2) admit solutions $\varphi_\lambda \in H$ which satisfy, for any $\lambda > 0$ small enough,

$$|\varphi_\lambda|_q \leqslant C|\lambda|^{\frac{2-b}{2(p-1)} - \frac{N}{2q}} \text{ if } 1 \leqslant q < \infty \text{ and } |\varphi_\lambda|_\infty \leqslant C|\lambda|^{\frac{2-b}{2(p-1)}}.$$

These decay estimates should be compared with the ones obtained in Theorem 5.9 of [21]. The comparison suggests that using a rescaling approach, as in the present paper, is fruitful to get the sharpest bifurcation estimates.

1.3 Stability

In this section we prove Theorem 1.2. The proof is divided into three steps. First we prove the convergence in H of the solutions $(\tilde{\varphi}_\lambda)$ of the rescaled problem to the unique positive solution $\psi \in H$ of the limit problem

$$-\Delta\varphi + \varphi = \frac{1}{|x|^b}|\varphi|^{p-1}\varphi, \ \varphi \in H. \tag{1.31}$$

Existence for (1.31) is standard because of the compactness of the nonlinear term and can, for example, be obtained by minimizing S under the constraint $I(v) = 0$ for $v \in H\backslash\{0\}$ where

$$S(v) = \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \frac{1}{|x|^b}|v(x)|^{p+1}dx, \tag{1.32}$$

$$I(v) = |v|_H^2 - \int_{\mathbb{R}^N} \frac{1}{|x|^b}|v(x)|^{p+1}dx. \tag{1.33}$$

We know from [11] that positive solutions of (1.31) are radial. They also decay exponentially at infinity. The uniqueness of $\psi \in H$ follows from [26].

Secondly, we establish some additional properties of the limit problem. In particular we prove that $\psi \in H$ is non degenerate.

In the third step, after having translated the stability criterion in the rescaled variables, we prove that it holds.

Notation Since in addition to (H1)-(H4) we now assume (H5)-(H7), we are somehow in the case of the modified problem, and therefore we will use the same notations. In particular, r will be now defined by

$$r(s) = g(s) - |s|^{p-1}s.$$

1.3.1 A convergence lemma

We start with a key technical result.

Lemma 1.10. *Assume (H1)-(H4). Let $(v_\lambda) \subset H$ be a bounded sequence in H and $q \in [1, p']$. Then we have, as $\lambda \to 0$,*

$$\int_{\mathbb{R}^N} \left|\frac{1}{|x|^b} - V_\lambda(x)\right| |v_\lambda(x)|^{q+1}dx \to 0.$$

Proof. For $R > 0$ we write

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\frac{1}{|x|^b} - V_\lambda(x)\right| |v_\lambda(x)|^{q+1}dx &\leqslant \int_{B(\sqrt{\lambda}R)} \left|\frac{1}{|x|^b} - V_\lambda(x)\right| |v_\lambda(x)|^{q+1}dx \\ &+ \int_{\mathbb{R}^N\backslash B(\sqrt{\lambda}R)} \left|\frac{1}{|x|^b} - V_\lambda(x)\right| |v_\lambda(x)|^{q+1}dx.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Fixing $R > 0$ large enough we have

$$\left|\frac{1}{|x|^b} - V_\lambda(x)\right| \leqslant \frac{\varepsilon}{|x|^b} \quad \text{for } x \in \mathbb{R}^N \backslash B(\sqrt{\lambda}R).$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^N \backslash B(\sqrt{\lambda}R)} \left|\frac{1}{|x|^b} - V_\lambda(x)\right| |v_\lambda(x)|^{q+1} dx \;&\leqslant\; \varepsilon \int_{B(1)\backslash B(\sqrt{\lambda}R)} \frac{1}{|x|^b} |v_\lambda(x)|^{q+1} dx \\ &+\; \varepsilon \int_{\mathbb{R}^N \backslash B(1)} |v_\lambda(x)|^{q+1} dx\end{aligned}$$

with, for $\theta = 2N/\{(N+2) - (N-2)q\}$,

$$\int_{B(1)\backslash B(\sqrt{\lambda}R)} \frac{1}{|x|^b} |v_\lambda(x)|^{q+1} dx \leqslant |\frac{1}{|x|^b}|_{L^\theta(B(1))} |v_\lambda|_{2^*}^{q+1} \leqslant C$$

and

$$\int_{\mathbb{R}^N \backslash B(1)} |v_\lambda(x)|^{q+1} dx \leqslant |v_\lambda|_{q+1}^{q+1} \leqslant C.$$

Now,

$$\begin{aligned}\int_{B(\sqrt{\lambda}R)} \left|\frac{1}{|x|^b} - V_\lambda(x)\right| |v_\lambda(x)|^{q+1} dx \;&\leqslant \\ &\left(|\frac{1}{|x|^b}|_{L^\theta(B(\sqrt{\lambda}R))} \;+\; |V_\lambda|_{L^\theta(B(\sqrt{\lambda}R))}\right) |v_\lambda|_{2^*}^{q+1}\end{aligned}$$

and since

$$|\frac{1}{|x|^b}|_{L^\theta(B(\sqrt{\lambda}R))} \to 0 \;\text{ and }\; |V_\lambda|_{L^\theta(B(\sqrt{\lambda}R))} = \lambda^{-b\theta/2+N/2} |V|_{L^\theta(B(R))} \to 0$$

as $\lambda \to 0$, this ends the proof. □

Now the main result of this subsection is

Lemma 1.11. *Assume (H1)-(H4). Then the solutions* $(\tilde{\varphi}_\lambda)_\lambda$ *of the rescaled equation (1.7) satisfy*

$$\lim_{\lambda \to 0} |\tilde{\varphi}_\lambda - \psi|_H = 0.$$

Proof. We divide the proof into two steps. First, we prove that there exists $(\mu(\lambda)) \subset \mathbb{R}$ such that $\mu(\lambda) \to 1$ and $(\mu(\lambda)\tilde{\varphi}_\lambda)$ is a minimizing sequence for

$$\min\{S(v), v \in H \setminus \{0\}, I(v) = 0\}. \tag{1.34}$$

Secondly, using this information, we prove the convergence of $(\tilde{\varphi}_\lambda)$ to ψ.

We begin by showing that $\limsup_{\lambda\to 0} S(\tilde{\varphi}_\lambda) \leqslant S(\psi)$. Let $\gamma_0 : [0,1] \to H$ be such that $\gamma_0(t) := Ct\psi$, for a $C > 0$. Then, fixing $C > 0$ large enough, we have $S(\gamma_0(1)) < 0$ and $S(\psi) = \max_{t\in[0,1]} S(\gamma_0(t))$ as it is easily seen from the simple "radial" behaviour of S.

Let $\varepsilon > 0$ be arbitrary. From Lemmas 1.5 and 1.10 we see that, for any $\lambda > 0$ small enough,

$$|\widehat{S}_\lambda(\gamma_0(s)) - S(\gamma_0(s))| \leqslant \varepsilon, \ \forall s \in [0,1]$$

and since $\widehat{S}_\lambda(\tilde{\varphi}_\lambda) = c(\lambda)$ it follows that

$$\tilde{S}_\lambda(\tilde{\varphi}_\lambda) = \widehat{S}_\lambda(\tilde{\varphi}_\lambda) \leqslant \max_{s\in[0,1]} \widehat{S}_\lambda(\gamma_0(s)) \leqslant \max_{s\in[0,1]} S(\gamma_0(s)) + \varepsilon = S(\psi) + \varepsilon.$$

Thus $\limsup\limits_{\lambda\to 0} \tilde{S}_\lambda(\tilde{\varphi}_\lambda) \leqslant S(\psi)$. Now, using Lemmas 1.5 and 1.10, we have

$$\lim_{\lambda\to 0} |S(\tilde{\varphi}_\lambda) - \tilde{S}_\lambda(\tilde{\varphi}_\lambda)| = 0$$

and we deduce that $\limsup\limits_{\lambda\to 0} S(\tilde{\varphi}_\lambda) \leqslant S(\psi)$.

Let us now show the existence of a sequence $(\mu(\lambda))$ such that $\mu(\lambda) \to 1$ and $I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0$. Since $\nabla\tilde{S}_\lambda(\tilde{\varphi}_\lambda)\tilde{\varphi}_\lambda = 0$ we have

$$I(\tilde{\varphi}_\lambda) = -\int_{\mathbb{R}^N} \left(\frac{1}{|x|^b} - V_\lambda(x)\right) |\tilde{\varphi}_\lambda|^{p+1} dx + \nabla\tilde{R}_\lambda(\tilde{\varphi}_\lambda)\tilde{\varphi}_\lambda.$$

Thus by Lemmas 1.5 and 1.10, $I(\tilde{\varphi}_\lambda) \to 0$. Let $\mu(\lambda) := \left(\dfrac{|\tilde{\varphi}_\lambda|_H^2}{\int_{\mathbb{R}^N} \frac{1}{|x|^b}|\tilde{\varphi}_\lambda|^{p+1}dx}\right)^{\frac{1}{p-1}}$.
Then $I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0$ and we have

$$|\mu(\lambda)^{p-1} - 1| = \frac{|I(\tilde{\varphi}_\lambda)|}{\int_{\mathbb{R}^N} \frac{1}{|x|^b}|\tilde{\varphi}_\lambda|^{p+1}dx}.$$

From the mountain pass geometry and since $\nabla\tilde{S}_\lambda(\tilde{\varphi}_\lambda)\tilde{\varphi}_\lambda = 0$ the denominator stays bounded away from 0 and since $I(\tilde{\varphi}_\lambda) \to 0$ we deduce that $\lim_{\lambda\to 0} \mu(\lambda) = 1$. Thus, by continuity of S, we have

$$\limsup_{\lambda\to 0} S(\mu(\lambda)\tilde{\varphi}_\lambda) = \limsup_{\lambda\to 0} S(\tilde{\varphi}_\lambda) \leqslant S(\psi)$$

and since $I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0$, $(\mu(\lambda)\tilde{\varphi}_\lambda)$ is a minimizing sequence for (1.34).

Now, using this information, we show the convergence of $(\tilde{\varphi}_\lambda)$ to ψ in H. Since $(\mu(\lambda)\tilde{\varphi}_\lambda)$ is bounded, there exists $\tilde{\varphi}_0$ such that, up to a subsequence, $\mu(\lambda)\tilde{\varphi}_\lambda \rightharpoonup \tilde{\varphi}_0$ weakly in H. Clearly, the minimizing sequences of (1.34) are the minimizing sequences of

$$\min\{|v|_H^2, v \in H \setminus \{0\}, I(v) = 0\},$$

and since for $v \in H$ such that $I(v) < 0$ there exists $0 < t < 1$ such that $I(tv) = 0$, (1.34) is also equivalent to

$$\min\{|v|_H^2, v \in H \setminus \{0\}, I(v) \leqslant 0\}.$$

If we assume that

$$|\tilde{\varphi}_0|_H^2 < \limsup_{\lambda \to 0} |\mu(\lambda)\tilde{\varphi}_\lambda|_H^2 = |\psi|_H^2 \tag{1.35}$$

since, as it can be prove in a standard way,

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} \frac{1}{|x|^b} |\mu(\lambda)\tilde{\varphi}_\lambda|^{p+1} dx = \int_{\mathbb{R}^N} \frac{1}{|x|^b} |\tilde{\varphi}_0|^{p+1} dx$$

we get that

$$I(\tilde{\varphi}_0) < \limsup_{\lambda \to 0} I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0.$$

Thus (1.35) contradicts the variational characterization of $\psi \in H$. We deduce that $\mu(\lambda)\tilde{\varphi}_\lambda \to \tilde{\varphi}_0$ strongly in H. In particular $\tilde{\varphi}_0$ is a minimizer of (1.34) and thus, by uniqueness, $\tilde{\varphi}_0 = \psi$. □

1.3.2 Further properties of the limit problem

We define the self adjoint operator $L_1 : D(L_1) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$L_1 = -\Delta + 1 - p\frac{1}{|x|^b}\psi^{p-1}$$

where $D(L_1) = \{v \in H^2(\mathbb{R}^N) : |x|^{-b}\psi^{p-1}v \in L^2(\mathbb{R}^N)\}$.

Proposition 1.12. *If $v \in D(L_1)$ satisfies $L_1 v = 0$ then $v = 0$.*

In the same spirit as Theorem 2.5 in [16], we performed a reduction of the problem by proving that the kernel of L_1 contains only radial functions.

Lemma 1.13. *If $v \in D(L_1)$ satisfies $L_1 v = 0$ then $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$.*

Before proving Lemma 1.13, we introduce some notations and recall some properties of spherical harmonics.

Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k with $\dim \mathcal{H}_k = a_k = \begin{pmatrix} k \\ N+k-1 \end{pmatrix} - \begin{pmatrix} k-2 \\ N+k-3 \end{pmatrix}$ for $k \geqslant 2$, $a_1 = N$, $a_0 = 1$. For each k let $\{Y_1^k, \dots, Y_{a_k}^k\}$ be an orthonormal basis of $\mathcal{H}_k$. It is known that any function $v \in L^2(\mathbb{R}^N)$ can be decomposed as follows

$$v = \sum_{k=0}^{+\infty} \sum_{i=1}^{a_k} v_{k,i}(|x|) Y_i^k \left(\frac{x}{|x|} \right)$$

where $v_{k,i}(r) := \displaystyle\int_{S^{N-1}} v(r\theta) Y_i^k(\theta) d\theta$.

Proof. Our proof follows a method due to [19] which has also been used in [15].

Let $v \in D(L_1)$ be such that $L_1 v = 0$ and consider its decomposition by spherical harmonics $\sum_{k=0}^{+\infty} \sum_{i=1}^{a_k} v_{k,i}(|x|) Y_i^k \left(\frac{x}{|x|} \right)$. Since $L_1 v = 0$, the functions $v_{k,i}$ satisfy

$$v_{k,i}'' + \frac{N-1}{r} v_{k,i}' + (-1 + \frac{p}{r^b} \psi^{p-1}) v_{k,i} - \frac{\mu_k}{r^2} v_{k,i} = 0 \tag{1.36}$$

where $\mu_k = k(k+N-2)$. It is standard to show that $v_{k,i} \in \mathcal{C}^2(0,+\infty)$, $\lim_{r\to 0} v_{k,i}(r)$ and $\lim_{r\to 0} r v_{k,i}'(r)$ exist and are finite, and both $v_{k,i}$ and $v_{k,i}'$ decay exponentially at infinity.

To prove the lemma it suffices to show that $v_{k,i} \equiv 0$, $\forall k \geqslant 1$.

The function $\psi(r) := \psi(|x|)$ satisfies

$$\psi'' + \frac{N-1}{r} \psi' - \psi + \frac{1}{r^b} \psi^p = 0, \tag{1.37}$$

thus $\psi \in \mathcal{C}^3(0,+\infty)$ and differentiating (1.37) we get

$$\psi''' + \frac{N-1}{r} \psi'' - \frac{N-1}{r^2} \psi' - \psi' + \frac{p}{r^b} \psi^{p-1} \psi' - \frac{b}{r^{b+1}} \psi^p = 0. \tag{1.38}$$

Let $0 < r_1 < r_2 < +\infty$. Multiplying (1.36) by $\psi' r^{N-1}$ and integrating over (r_1, r_2) it follows that

$$\int_{r_1}^{r_2} v_{k,i} r^{N-1} (\psi''' + \frac{N-1}{r} \psi'' - \psi' + \frac{p}{r^b} \psi^{p-1} \psi') - \mu_k v_{k,i} r^{N-3} \psi' dr + g(r_2) - g(r_1) = 0$$

where $g(r) := \psi' r^{N-1} v'_{k,i} - \psi'' r^{N-1} v_{k,i}$. Using (1.38), we get

$$(N-1-\mu_k)\int_{r_1}^{r_2} v_{k,i} r^{N-3}\psi' dr + \int_{r_1}^{r_2} v_{k,i} r^{N-1}\frac{b}{r^{b+1}}\psi^p dr + g(r_2) - g(r_1) = 0. \quad (1.39)$$

Because ψ', ψ'' decay exponentially at infinity (see the Appendix) we have $g(r) \to 0$ as $r \to +\infty$. Since $N \geqslant 3$ we also have $g(r) \to 0$ as $r \to 0$.

Arguing by contradiction, we suppose $v_{k,i} \not\equiv 0$. Then, considering $-v_{k,i}$ instead of $v_{k,i}$ if necessary, there exist $0 \leqslant \alpha < \beta \leqslant +\infty$ such that

(i) $v_{k,i}(r) > 0$ in (α, β),

(ii) $v_{k,i}(\alpha) = 0$ if $\alpha \neq 0$ and $v_{k,i}(\beta) = 0$ if $\beta \neq +\infty$,

(iii) $v'_{k,i}(\alpha) \geqslant 0$ if $\alpha \neq 0$ and $v'_{k,i}(\beta) \leqslant 0$ if $\beta \neq +\infty$.

It is standard to show that $\psi' < 0$ (see [11]), thus we have $g(\alpha) \leqslant 0$ and $g(\beta) \geqslant 0$. Therefore $g(\beta) - g(\alpha) \geqslant 0$ and thanks to (1.39) we have

$$(N-1-\mu_k)\int_a^b v_{k,i} r^{N-3}\psi' dr + \int_a^b v_{k,i} r^{N-1}\frac{b}{r^{b+1}}\psi^b \leqslant 0.$$

However, since $\psi' < 0$ and $N-1-\mu_k \leqslant 0$, we should have

$$(N-1-\mu_k)\int_a^b v_{k,i} r^{N-3}\psi' dr + \int_a^b v_{k,i} r^{N-1}\frac{b}{r^{b+1}}\psi^b > 0.$$

This contradiction proves that $v_{k,i} \equiv 0$ for all $k \geqslant 1$. □

We are now in position to prove Proposition 1.12

Proof of Proposition 1.12. Our proof borrows some elements from [15] and [16]. Thanks to Lemma 1.13, it is enough to prove Proposition 1.12 for radial functions, therefore we work in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

For $\delta > 0$ small, we consider the following perturbation of (1.31)

$$-\Delta v + (1 + \delta e^{-|x|^{-1}-|x|}\psi^{p-1})v = \left(\frac{1}{|x|^b} + \delta e^{-|x|^{-1}-|x|}\right) v_+^p,\ v \in H^1_{\mathrm{rad}}(\mathbb{R}^N). \quad (1.40)$$

Solutions of (1.40) are positive and can be obtained by minimizing the functional S_δ under the natural constraint $I_\delta(v) = 0$ for $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, where

$$\begin{aligned}
S_\delta(v) &= \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \frac{1}{|x|^b} v_+^{p+1} dx \\
&\quad -\delta\left(\frac{1}{p+1}\int_{\mathbb{R}^N} e^{-|x|^{-1}-|x|} v_+^{p+1} dx - \frac{1}{2}\int_{\mathbb{R}^N} e^{-|x|^{-1}-|x|}\psi^{p-1} v^2 dx\right), \\
I_\delta(v) &= |v|_H^2 - \int_{\mathbb{R}^N} \frac{1}{|x|^b} v_+^{p+1} dx \\
&\quad -\delta\left(\int_{\mathbb{R}^N} e^{-|x|^{-1}-|x|} v_+^{p+1} dx - \int_{\mathbb{R}^N} e^{-|x|^{-1}-|x|}\psi^{p-1} v^2 dx\right).
\end{aligned}$$

Here both S_δ and I_δ are defined on $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and it is standard to show that they are of class $\mathcal{C}^2$.

We shall see in the Appendix that (1.40) has a unique positive radial solution for $\delta > 0$ small, and since $\psi \in H$ satisfies (1.40), it is this unique solution. In particular, $\psi \in H$ solves

minimize $S_\delta(v)$ under the constraint $I_\delta(v) = 0$ for $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$.

We recall that the Morse index of S_δ at ψ is given by

$$\begin{aligned}
\mathrm{Index}\, S''_\delta(\psi) &= \max\{\dim V : V \subset H^1_{\mathrm{rad}}(\mathbb{R}^N) \text{ is a subspace such that} \\
&\quad \langle S''_\delta(\psi)h, h\rangle < 0 \text{ for all h } \in V \setminus \{0\}\}.
\end{aligned}$$

We claim that Index $S''_\delta(\psi) \leqslant 1$. To see this let us show that $\langle S''_\delta(\psi)v, v\rangle \geqslant 0$ on the subspace of co-dimension one $\{v \in H \mid \nabla I_\delta(\psi)v = 0\}$.

Let $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be such that $\nabla I_\delta(\psi)v = 0$. Using the Implicit function theorem, we see that there exist $\varepsilon > 0$ and a $\mathcal{C}^2$-curve $\phi : (-\varepsilon, \varepsilon) \to H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$\phi(0) = \psi,\ \phi'(0) = v \text{ and } I_\delta(\phi(t)) = 0.$$

Thanks to the variational characterization of ψ, 0 is a local minimum of $t \mapsto S_\delta(\phi(t))$, and therefore $\frac{d^2}{dt^2}S_\delta(\phi(t))|_{t=0} \geqslant 0$. But, since $\nabla S_\delta(\psi) = 0$, we have

$$0 \leqslant \frac{d^2}{dt^2}S_\delta(\phi(t))|_{t=0} = \langle S''_\delta(\psi)v, v\rangle\,.$$

At this point our claim is establish. Now seeking a contradiction we assume the existence of $v_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$ such that $L_1 v_0 = 0$. Let $V := \mathrm{span}\{v_0, \psi\}$. Since

$$\langle L_1\psi, \psi\rangle = -(p-1)\int_{\mathbb{R}^N} \frac{1}{|x|^b}\psi^{p+1} dx < 0$$

and $\langle L_1 v_0, v\rangle = 0$ for all $v \in H^1_{\text{rad}}(\mathbb{R}^N)$, we see that V is of dimension 2 and that, for all $h \in V$, $\langle L_1 h, h\rangle \leqslant 0$. Thus we have, for all $h \in V \setminus \{0\}$,

$$\langle S''_\delta(\psi)h, h\rangle \;=\; \langle L_1 h, h\rangle - \delta(p-1)\int_{\mathbb{R}^N} \psi^{p-1}h^2 dx < 0$$

which implies that Index $S''_\delta(\psi) \geqslant 2$. This contradiction ends the proof. □

Lemma 1.14. *[Spectral properties] The spectrum $\sigma(L_1)$ of L_1 contains a simple first eigenvalue $-\lambda_1 < 0$ and $\sigma(L_1) \setminus \{\lambda_1\} \subset (0, +\infty)$. Thus if $e_1 \in H$ denote an eigenvector associated to $-\lambda_1$, such that $|e_1|_2 = 1$, then H can be decomposed as $H = E_1 \oplus E_+$ where $E_1 = \operatorname{span}\{e_1\}$, E_+ is the eigenspace corresponding to the positive part of $\sigma(L_1)$ restricted to H and $E_1 \perp E_+$ (where $\perp$ denote the orthogonality in $L^2(\mathbb{R}^N)$).*

Proof. Since $\langle L_1\psi, \psi\rangle < 0$, the first eigenvalue $-\lambda_1$ is negative, and it is standard to show that $-\lambda_1$ is simple. From Weyl's theorem, we see that the essential spectrum of L_1 is in $[1, +\infty)$ and that the spectrum in $(-\lambda_1, \frac{1}{2}]$ contains only a finite number of eigenvalues. Thanks to Proposition 1.12, the null-space of L_1 is empty. Therefore to prove the lemma it just remains to show that $\lambda_2 \geqslant 0$ if it exists.

Arguing by contradiction, we suppose that the second eigenvalue is $-\lambda_2 < 0$ with an associated eigenvector e_2 and $|e_2|_2 = 1$. Since L_1 is selfadjoint, we have $(e_1, e_2)_2 = 0$. Let $\mu, \nu \in \mathbb{R}$. We have

$$\langle L_1(\mu e_1 + \nu e_2), \mu e_1 + \nu e_2\rangle = -\lambda_1\mu^2 - \lambda_2\nu^2 < 0.$$

In other words, L_1 is negative on a subspace of dimension 2. But, arguing as in Proposition 1.12, we can prove that L_1 is nonnegative on the subspace $\{v \in H \,|\, \nabla I(\psi)v = 0\}$ of codimension 1, raising a contradiction. □

Lemma 1.15. *If $v \in H$ satisfies $(v, \psi)_2 = 0$ and $\langle L_1 v, v\rangle \leqslant 0$, then $v \equiv 0$. Here $(\cdot,\cdot)_2$ is the standard scalar product on $L^2(\mathbb{R}^N)$.*

Proof. We introduce $\psi_\lambda := \lambda^{\frac{2-b}{2(p-1)}}\psi(\sqrt{\lambda}x)$. Since ψ is solution of (1.31), $\psi_\lambda \in H$ satisfies

$$-\Delta\psi_\lambda + \lambda\psi_\lambda - \frac{1}{|x|^b}\psi_\lambda^p = 0. \tag{1.41}$$

Differentiating (1.41) with respect to λ gives for $\lambda = 1$

$$-\Delta w + w - \frac{p}{|x|^b}\psi^{p-1}w = -\psi \;\text{ where }\; w = \frac{2-b}{2(p-1)}\psi + \frac{1}{2}x\cdot\nabla\psi. \tag{1.42}$$

Namely $L_1 w = -\psi$.

Let $v \in H$ be such that $v \not\equiv 0$ and $(v, \psi)_2 = 0$. To prove Lemma 1.15 it suffices to show that $\langle L_1 v, v\rangle > 0$.

Using the orthogonal spectral decomposition $H = E_1 \oplus E_+$ we write v and w as

$$\begin{array}{rcl} v &=& \alpha e_1 + \xi \\ w &=& \beta e_1 + \zeta \end{array} \text{ where } \xi, \zeta \in E_+.$$

Therefore we have

$$\begin{array}{rcl} \langle L_1 v, v\rangle &=& -\alpha^2\lambda_1 + \langle L_1\xi, \xi\rangle \\ \langle L_1 w, w\rangle &=& -\beta^2\lambda_1 + \langle L_1\zeta, \zeta\rangle\,. \end{array} \tag{1.43}$$

If $\alpha = 0$, then $\xi \not\equiv 0$ and $\langle L_1 v, v\rangle > 0$ is satisfied. In the sequel, we suppose $\alpha \neq 0$. From the expression of w, we have

$$\langle L_1 w, w\rangle = -\frac{1}{2}\left(\frac{2-b}{p-1} - \frac{N}{2}\right)|\psi|_2^2 < 0. \tag{1.44}$$

Also from (1.42) and $(v, \psi)_2 = 0$, it follows that

$$0 = (\psi, v)_2 = \langle L_1 w, v\rangle = -\alpha\beta\lambda_1 + \langle L_1\zeta, \xi\rangle$$

and therefore

$$\langle L_1\zeta, \xi\rangle = \alpha\beta\lambda_1. \tag{1.45}$$

Consequently, $\zeta \not\equiv 0$ since otherwise (1.45) would give $\beta = 0$, which leads to a contradiction in (1.44). Since $L_1 > 0$ on E_+, the inequality $\langle L_1\zeta, \xi\rangle^2 \leqslant \langle L_1\zeta, \zeta\rangle\langle L_1\xi, \xi\rangle$ holds. Combining (1.42)–(1.44) we obtain

$$\begin{array}{rcl} \langle L_1 v, v\rangle = -\alpha^2\lambda_1 &+& \langle L_1\xi, \xi\rangle \geqslant -\alpha^2\lambda_1 + \dfrac{\langle L_1\xi, \zeta\rangle^2}{\langle L_1\zeta, \zeta\rangle} \\ &=& -\alpha^2\lambda_1 + \dfrac{\alpha^2\beta^2\lambda_1^2}{\beta^2\lambda_1 + \langle L_1 w, w\rangle} \\ &=& \dfrac{-\langle L_1 w, w\rangle\,\alpha^2\lambda_1}{\langle L_1\zeta, \zeta\rangle} > 0. \end{array}$$

This ends the proof. □

Remark 1.16. Our proof of Lemma 1.15 is inspired by the work [13], which was indicated to us by R. Fukuizumi. In Lemma 2.1 of [6] (see also Proposition 2.7 of [25]) an alternative proof of Lemma 1.15 is given. Another proof of Lemma 1.15 relying on the fact that ψ is a local minimum of S on the sphere of corresponding L^2-norm can also be performed [17].

1.3.3 Verification of the stability criterion

To prove Theorem 1.2 we shall use Proposition 1.3. Since the convergence result holds in the rescaled variables it is convenient to express Proposition 1.3 in these variables. For $v \in H^1(\mathbb{R}^N, \mathbb{C})$, let $\tilde{v} \in H^1(\mathbb{R}^N, \mathbb{C})$ be defined by

$$v(x) = \lambda^{\frac{2-b}{2(p-1)}} \tilde{v}(\sqrt{\lambda} x).$$

Then we have

$$\begin{aligned} \langle S''_\lambda(\varphi_\lambda) v, v \rangle &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}} \left\langle \tilde{S}''_\lambda(\tilde{\varphi}_\lambda)\tilde{v}, \tilde{v} \right\rangle, \\ \|\nabla v\|_2^2 + \lambda \|v\|_2^2 &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}} \|\tilde{v}\|_2^2, \\ \langle \varphi_\lambda, v \rangle_2 &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}} \langle \tilde{\varphi}_\lambda, \tilde{v} \rangle_2, \\ \langle i\varphi_\lambda, v \rangle_2 &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}} \langle i\tilde{\varphi}_\lambda, \tilde{v} \rangle_2, \end{aligned}$$

where now by $\tilde{S}_\lambda$ we denote the extension of $\tilde{S}_\lambda$ from H to $H^1(\mathbb{R}^N;\mathbb{C})$. Therefore, if there exists $\delta > 0$ such that for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfying $\langle \tilde{\varphi}_\lambda, \tilde{v} \rangle_2 = \langle i\tilde{\varphi}_\lambda, \tilde{v} \rangle_2 = 0$ we have

$$\left\langle \tilde{S}''_\lambda(\tilde{\varphi}_\lambda)\tilde{v}, \tilde{v} \right\rangle \geqslant \delta ||\tilde{v}||^2, \tag{1.46}$$

we have, for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfying $\langle \varphi_\lambda, v \rangle_2 = \langle i\varphi_\lambda, v \rangle_2 = 0$,

$$\langle S''_\lambda(\varphi_\lambda) v, v \rangle \geqslant \delta(\|\nabla v\|_2^2 + \lambda \|v\|_2^2). \tag{1.47}$$

Clearly, for $v \in H^1(\mathbb{R}^N, \mathbb{C})$ the norm $\sqrt{\|\nabla v\|_2^2 + \lambda \|v\|_2^2}$ is equivalent to the norm $||v||$ and thus proving (1.46) suffices to check the assumptions of Proposition 1.3.

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$, let $v_1 = \mathrm{Re} v$ and $v_2 = \mathrm{Im} v$. Then we have, after some calculations,

$$\left\langle \tilde{S}''_\lambda(\tilde{\varphi}_\lambda) v, v \right\rangle = \left\langle \tilde{L}_{1,\lambda} v_1, v_1 \right\rangle + \left\langle \tilde{L}_{2,\lambda} v_2, v_2 \right\rangle,$$

with

$$\begin{aligned} \left\langle \tilde{L}_{1,\lambda} v_1, v_1 \right\rangle &= |v_1|_H^2 - p \int_{\mathbb{R}^N} V_\lambda(x) \tilde{\varphi}_\lambda^{p-1} |v_1|^2 dx \\ &\quad - \int_{\mathbb{R}^N} V_\lambda(x) \lambda^{-1+\frac{b}{2}} r' \left(\lambda^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_\lambda \right) |v_1|^2 dx, \\ \left\langle \tilde{L}_{2,\lambda} v_2, v_2 \right\rangle &= |v_2|_H^2 - \int_{\mathbb{R}^N} V_\lambda(x) \tilde{\varphi}_\lambda^{p-1} |v_2|^2 dx \\ &\quad - \int_{\mathbb{R}^N} V_\lambda(x) \lambda^{\frac{b}{2}} \left(\frac{\tilde{r}(\tilde{\varphi}_\lambda(x))}{\tilde{\varphi}_\lambda(x)} \right) |v_2|^2 dx. \end{aligned}$$

In addition $\langle \tilde{\varphi}_\lambda, v \rangle_2 = (\tilde{\varphi}_\lambda, v_1)_2$ et $\langle i\tilde{\varphi}_\lambda, v \rangle_2 = (\tilde{\varphi}_\lambda, v_2)_2$. Thus, to ends the proof of Theorem 1.2 it is enough to prove the following lemma.

Lemma 1.17. *Assume (H1)-(H7). There exists* $\tilde{\lambda} > 0$ *such that*

(i) there exists $\delta_1 > 0$ *such that* $\left\langle \tilde{L}_{1,\lambda} v, v \right\rangle \geqslant \delta_1 |v|_H^2$ *for all* $v \in H$ *satisfying* $(v, \tilde{\varphi}_\lambda)_2 = 0$*, for all* $\lambda \in (0, \tilde{\lambda}]$*;*

(ii) there exists $\delta_2 > 0$ *such that* $\left\langle \tilde{L}_{2,\lambda} v, v \right\rangle \geqslant \delta_2 |v|_H^2$ *for all* $v \in H$ *satisfying* $(v, \tilde{\varphi}_\lambda)_2 = 0$*, for all* $\lambda \in (0, \tilde{\lambda}]$*.*

Proof. Seeking a contradiction for part (i), we assume that there exist $(\lambda_j) \subset \mathbb{R}^+$ with $\lambda_j \to 0$ and $(v_j) \in H$ such that

$$\begin{aligned} &\lim_{j\to\infty} \left\langle \tilde{L}_{1,\lambda_j} v_j, v_j \right\rangle \leqslant 0, \\ &|v_j|_H = 1, \qquad (v_j, \tilde{\varphi}_{\lambda_j})_2 = 0. \end{aligned}$$

Since $(v_j) \subset H$ is bounded, there exists $v_\infty \in H$ such that $v_j \rightharpoonup v_\infty$ weakly in H. Let us prove that

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} V_{\lambda_j}(x) \lambda_j^{-1+\frac{b}{2}} r' \left(\lambda_j^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_{\lambda_j} \right) |v_j|^2 dx = 0, \tag{1.48}$$

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} V_{\lambda_j}(x) \tilde{\varphi}_{\lambda_j}^{p-1} |v_j|^2 dx = \int_{\mathbb{R}^N} \frac{1}{|x|^b} \psi^{p-1} |v_\infty|^2 dx. \tag{1.49}$$

To prove (1.48) let $\varepsilon > 0$ be arbitrary. By (H7), we have $\lim_{s\to 0^+} \dfrac{r'(s)}{s^{p-1}} = 0$. Moreover, $(|\tilde{\varphi}_{\lambda_j}|_\infty)$ is bounded and therefore, for any $\lambda > 0$ sufficiently small, $r'\left(\lambda_j^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_{\lambda_j} \right) \leqslant C\varepsilon \lambda_j^{1-\frac{b}{2}}$. Thus

$$\left| \int_{\mathbb{R}^N} V_{\lambda_j}(x) \lambda_j^{-1+\frac{b}{2}} r' \left(\lambda_j^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_{\lambda_j} \right) |v_j|^2 dx \right| \leqslant \varepsilon C \left| \int_{\mathbb{R}^N} V_{\lambda_j}(x) |v_j|^2 dx \right|$$

and we conclude by Lemma 1.4. Clearly proving (1.49) is equivalent to show that, as $\lambda \to 0$,

$$\int_{\mathbb{R}^N} \left(V_{\lambda_j}(x) - \frac{1}{|x|^b} \right) \tilde{\varphi}_{\lambda_j}^{p-1} |v_j|^2 dx \to 0, \tag{1.50}$$

$$\int_{\mathbb{R}^N} \frac{1}{|x|^b} \left(\tilde{\varphi}_{\lambda_j}^{p-1} |v_j|^2 - \psi^{p-1} |v_\infty|^2 \right) dx \to 0. \tag{1.51}$$

Since $(|\tilde{\varphi}_{\lambda_j}|_\infty)$ is bounded, Lemma 1.10 shows that (1.50) holds. Now since $|x|^{-b} \to 0$ as $|x| \to \infty$ to show (1.51) it suffices to show that, $\forall R > 0$,

$$\int_{B(R)} \frac{1}{|x|^b} \left(\tilde{\varphi}_{\lambda_j}^{p-1} |v_j|^2 - \psi^{p-1} |v_\infty|^2 \right) dx \to 0. \tag{1.52}$$

We write

$$\int_{B(R)} \frac{1}{|x|^b} \tilde{\varphi}_{\lambda_j}^{p-1} |v_j|^2 dx = \int_{B(R)} \frac{1}{|x|^b} (\tilde{\varphi}_{\lambda_j}^{p-1} - \psi^{p-1}) |v_j|^2 dx + \int_{B(R)} \frac{1}{|x|^b} \psi^{p-1} |v_j|^2 dx.$$

Since $\tilde{\varphi}_{\lambda_j} \to \psi$ in H, we have, up to a subsequence, $|x|^{-b}\tilde{\varphi}_{\lambda_j}^{p-1} \to |x|^{-b}\psi^{p-1}$ a.e. and since

$$||x|^{-b}\tilde{\varphi}_{\lambda_j}^{p-1}| \leqslant C|x|^{-b} \in L^{\frac{N}{2}}(B(R)),$$

Lebesgue's Theorem gives $|x|^{-b}\tilde{\varphi}_{\lambda_j}^{p-1} \to |x|^{-b}\psi^{p-1}$ in $L^{\frac{N}{2}}(B(R))$. Also we have $|v_j|^2 \rightharpoonup |v_\infty|^2$ weakly in $L^{\frac{N}{N-2}}(B(R))$. At this point (1.52) follows easily.

Now, on one hand, from (1.48)-(1.49) we have

$$\lim_{j\to\infty} \left\langle \tilde{L}_{1,\lambda_j} v_j, v_j \right\rangle = 1 - p \int_{\mathbb{R}^N} \frac{1}{|x|^b} \psi^{p-1} |v_\infty|^2 dx. \tag{1.53}$$

On the other hand, still by (1.48)-(1.49) and the weak convergence $v_j \rightharpoonup v_\infty$ in H we have $(v_\infty, \psi)_2 = 0$ and,

$$\langle L_1 v_\infty, v_\infty \rangle \leqslant \lim_{j\to\infty} \left\langle \tilde{L}_{1,\lambda_j} v_j, v_j \right\rangle \leqslant 0 \text{ (by assumption)}$$

which implies, according to Lemma 1.15, that $v_\infty \equiv 0$. But this leads to a contradiction in (1.53) and finishes the proof of (i). To prove (ii), since (i) holds, it suffices to show that, for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |V_\lambda(x)| \lambda^{\frac{b}{2}} \left(\frac{\tilde{r}(\tilde{\varphi}_\lambda)}{\tilde{\varphi}_\lambda} \right) |v|^2 dx \leqslant \varepsilon$$

when $|v|_H = 1$ and $\lambda > 0$ is sufficiently small. Let $\varepsilon > 0$ be arbitrary. Since $(|\tilde{\varphi}_\lambda|_\infty)$ is bounded, for $\lambda > 0$ small enough, we have from (1.8) that

$$\frac{\tilde{r}(\tilde{\varphi}_\lambda)}{\tilde{\varphi}_\lambda} \leqslant \varepsilon \lambda_j^{-\frac{b}{2}} |\tilde{\varphi}_\lambda|^{p-1}.$$

Thus

$$\int_{\mathbb{R}^N} |V_\lambda(x)| \lambda^{\frac{b}{2}} \left(\frac{\tilde{r}(\tilde{\varphi}_\lambda)}{\tilde{\varphi}_\lambda} \right) |v|^2 dx \leqslant \varepsilon C \int_{\mathbb{R}^N} |V_\lambda(x)| |v|^2 dx \leqslant \varepsilon C$$

by Lemma 1.4 and we conclude. □

1.4 Appendix

Here, we prove the uniqueness of the non-zero solutions of (1.40). For this we use results of [26].

It is known that solutions v of (1.40) are in $\mathcal{C}(\mathbb{R}^N) \cap \mathcal{C}^2(\mathbb{R}^N \setminus \{0\})$ and decay exponentially at infinity. Also setting $v = v(r)$, $r = |x|$, we have $\lim_{r\to 0} rv_r(r) = 0$ (where $v_r = \frac{\partial v}{\partial r}$) and v satisfies the ordinary differential equation

$$v_{rr} + \frac{N-1}{r} v_r + g(r)v + h(r)v_+^p = 0 \tag{1.54}$$

where $g(r) = -(1 + \delta e^{-r^{-1}-r}\psi(r)^{p-1})$ and $h(r) = r^{-b} + \delta e^{-r^{-1}-r}$. For $m \in [0, N-2]$ we define

$$\begin{aligned} G(r,m) &= -r^{m+2}\delta f_r - \alpha_1 r^{m+1}(1+\delta f) + \alpha_2 r^{m-1}, \\ H(r,m) &= -\left(\beta + \frac{2b}{p+1}\right) r^{m+1-b} - \frac{2\delta}{p+1} r^m (r^2-1) e^{-r^{-1}-r} - \beta r^{m+1}\delta e^{-r^{-1}-r}, \end{aligned}$$

where $f := e^{-r^{-1}-r}\psi^{p-1}$, $\alpha_1 := -2(N-3-m)$, $\alpha_2 := m(N-2-m)(2N-4-m)/2$ and $\beta := 2N-4-m-2(m+2)/(p+1)$.

According to Theorem 2.2 of [26] to establish the uniqueness of the positive solution of (1.54) it suffices to check the following conditions.

- (A1) g and h are in $\mathcal{C}^1((0,\infty))$,
- (A2) $r^{2-\sigma}g(r) \to 0$ and $r^{2-\sigma}h(r) \to 0$ as $r \to 0^+$ for some $\sigma > 0$,
- (C1) $h(r) \geqslant 0$ for all $r \in (0,\infty)$ and there exists $r_0 > 0$ such that $h(r_0) > 0$,
- (C2) $G(r, N-2) \leqslant 0$ for all $r \in (0,\infty)$,
- (C3) for each $m \in [0, N-2)$, there exists $\alpha(m) \in [0,\infty]$ such that $G(r,m) \geqslant 0$ for $r \in (0, \alpha(m))$ and $G(r,m) \leqslant 0$ for $r \in (\alpha(m), \infty)$,
- (C4) $H(r,0) \leqslant 0$ for all $r \in (0,\infty)$,
- (C5) for each $m \in (0, N-2]$, there exists $\beta(m) \in [0,\infty)$ such that $H(r,m) \geqslant 0$ for $r \in (0,\beta(m))$ and $H(r,m) \leqslant 0$ for $r \in (\beta(m),\infty)$.

In (C3), by $\alpha(m) = 0$ and $\alpha(m) = \infty$ we mean that $G(s,m) \leqslant 0$ and $G(s,m) \geqslant 0$, respectively, for all $s \in (0,\infty)$. The analogous convention holds for (C5).

The following lemma is useful to check (C1)-(C5). It was provided to us by K. Tanaka [23].

Lemma 1.18. *Let* $f(r) = e^{-r^{-1}-r}\psi(r)^{p-1}$*. Then* $f(r)$*,* $f_r(r)$ *and* $f_{rr}(r)$ *are bounded on* $(0, +\infty)$ *and exponentially decaying at infinity.*

Proof. First, we prove that there exist constants $R_0 > 0$ and $C > 0$ such that

$$0 \leqslant -\psi_r(r) \leqslant C_2\psi(r) \ \text{ for all } r \in [R_0, \infty). \tag{1.55}$$

Let $W(r) = 1 - r^{-b}\psi(r)^{p-1}$. Then $\psi(r)$ satisfies

$$-\psi_{rr}(r) - \frac{N-1}{r}\psi_r(r) + W(r)\psi(r) = 0 \tag{1.56}$$

and defining $R(r)$ and $\theta(r)$ by

$$\begin{aligned} r^{N-1}\psi(r) &= R(r)\sin\theta(r), \\ r^{N-1}\psi_r(r) &= R(r)\cos\theta(r) \end{aligned}$$

it follows that $\theta(r)$ verifies

$$\theta_r(r) = \cos^2\theta(r) - W(r)\sin^2\theta(r) + \frac{N-1}{r}\sin\theta(r)\cos\theta(r). \tag{1.57}$$

It is standard (see [11]) that $\psi_r(r) < 0$, $\forall r \in (0, \infty)$. Thus $\theta(r) \subset [\pi/2, \pi]$. In addition, since $W(r) \to 1$ as $r \to \infty$, the right hand side of (1.57) is negative in a neighbourhood of $\pi/2^+$ and positive in a neighbourhood of π^-, for $r > 0$ sufficiently large. This shows that $\theta(r)$ stays, for $r > 0$ large, confined in a interval $[a, b] \subset (\pi/2, \pi)$. This implies (1.55). Now we have, for $r > 0$ large,

$$\left|\frac{\partial}{\partial r}\psi(r)^{p-1}\right| = (p-1)\psi(r)^{p-2}|\psi_r(r)| \leqslant (p-1)C\psi(r)^{p-1},$$

and we can easily deduce that $f_r(r)$ is exponentially decaying. Also, we have

$$\frac{\partial^2}{\partial r^2}\psi(r)^{p-1} = (p-1)\psi(r)^{p-2}\psi_{rr}(r) + (p-1)(p-2)\psi(r)^{p-3}\psi_r(r)^2.$$

The term $(p-1)(p-2)\psi(r)^{p-3}\psi_r(r)^2$ can be treated as previously and thanks (1.56) we have

$$\psi(r)^{p-2}\psi_{rr}(r) = -\frac{N-1}{r}\psi(r)^{p-2}\psi_r(r) + W(r)\psi(r)^{p-1},$$

which allows us to conclude that $f_{rr}(r)$ is also exponentially decaying.

Finally, since $\psi \in \mathcal{C}([0, +\infty)) \cap \mathcal{C}^2((0, +\infty))$ and $\lim_{r\to 0} r\psi_r(r) = 0$, it is clear that $f(r)$ and $f_r(r)$ are bounded on $(0, +\infty)$, and using the equation for ψ, we also see that $f_{rr}(r)$ is bounded on $(0, +\infty)$. □

The conditions (A1), (A2) and (C1) are clearly satisfied. For (C2), we have

$$G(r, N-2) = -r^{N-1}(\delta(rf_r(r) + 2f(r)) + 2).$$

Thanks to Lemma 1.18, $t \mapsto (rf_r(r) + 2f(r))$ is bounded on $(0, +\infty)$, therefore, for $\delta > 0$ small enough (C2) is verified. For (C3), we distinguish two cases. If $N - 3 - m > 0$, then $\alpha_1 < 0$, $\alpha_2 > 0$ and we have

$$G(r, m) = r^{m+1}(-r\delta f_r(r) - \alpha_1\delta f(r) - \alpha_1) + \alpha_2 r^{m-1}.$$

Thanks to Lemma 1.18, $-r\delta f_r(r) - \alpha_1\delta f(r) - \alpha_1 > 0$ for $\delta > 0$ small enough, and consequently $G(r, m) \geqslant 0$ for all $r \in (0, \infty)$. If $N - 3 - m \leqslant 0$ then $\alpha_1 \geqslant 0$, $\alpha_2 > 0$ and thus we have

$$\frac{\partial}{\partial r}\left(\frac{G(r, m)}{r^{m+1}}\right) = -\delta f_r(r) - r\delta f_{rr}(r) - \alpha_1\delta f_r(r) - 2\alpha_2 r^{-3} < 0$$

for $\delta > 0$ sufficiently small. Thus (C3) also hold. Now

$$H(r, 0) = -\left(\beta + \frac{2b}{p+1}\right) r^{1-b} + \frac{2\delta}{p+1} e^{-r^{-1}-r} - \frac{2\delta}{p+1} r^2 e^{-r^{-1}-r} - \beta r\delta e^{-r^{-1}-r}.$$

We remark that $\beta > 0$ and that, for δ small enough,

$$\frac{2\delta}{p+1} e^{-r^{-1}-r} < \left(\beta + \frac{2b}{p+1}\right) r^{1-b},$$

thus we see that (C4) holds. Let $m \in (0, N-2]$. We have

$$\frac{H(r, m)}{r^{m+1-b}} = -\left(\beta + \frac{2b}{p+1}\right) - \delta\left(\frac{2(r - r^{-1})}{p+1} + \beta\right) r^b e^{-r^{-1}-r}.$$

Since the function $r \mapsto [2(r - r^{-1})/(p+1) + \beta] r^b e^{-r^{-1}-r}$ is bounded, when $\beta + 2b/(p+1) \neq 0$ the sign of $H(r, m)$ is constant for $\delta > 0$ small enough. When $\beta + 2b/(p+1) = 0$ we see that there exists $\beta(m) := (-b + \sqrt{b^2 + 4})/2$ such that the function $r \to -\frac{2\delta}{p+1}(r^2 + b - 1) r^{b-1} e^{-r^{-1}-r}$ is positive on $(0, \beta(m))$ and negative on $(\beta(m), \infty)$. Therefore, in both cases $H(r, m)$ satisfies (C5).

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Chapitre 2

Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential

Abstract. We study analytically and numerically the stability of the standing waves for a nonlinear Schrödinger equation with a point defect and a power type nonlinearity. A main difficulty is to compute the number of negative eigenvalues of the linearized operator around the standing waves, and it is overcome by a perturbation method and continuation arguments. Among others, in the case of a repulsive defect, we show that the standing wave solution is stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ and unstable in $H^1(\mathbb{R})$ under subcritical nonlinearity. Further we investigate the nature of instability: under critical or supercritical nonlinear interaction, we prove the instability by blow-up in the repulsive case by showing a virial theorem and using a minimization method involving two constraints. In the subcritical radial case, unstable bound states cannot collapse, but rather narrow down until they reach the stable regime (a *finite-width instability*). In the non-radial repulsive case, all bound states are unstable, and the instability is manifested by a lateral drift away from the defect, sometimes in combination with a finite-width instability or a blowup instability.

2.1 Introduction

We consider a nonlinear Schrödinger equation with a delta function potential

$$\begin{cases} i\partial_t u(t,x) = -\partial_{xx}u - \gamma u\delta(x) - |u|^{p-1}u, \\ u(0,x) = u_0, \end{cases} \tag{2.1}$$

where $\gamma \in \mathbb{R}$, $1 < p < +\infty$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Here, δ is the Dirac distribution at the origin. Namely, $\langle \delta, v \rangle = v(0)$ for $v \in H^1(\mathbb{R})$.

When $\gamma = 0$, this type of equations arises in various physical situations in the description of nonlinear waves (see [36] and the references therein); especially in nonlinear optics, it describes the propagation of a laser beam in a homogeneous medium. When $\gamma \neq 0$, equation (2.1) models the nonlinear propagation of light through optical waveguides with a localized defect (see [5, 18, 21, 29] and the references therein for more detailed considerations on the physical background). The authors in [5, 18, 21, 22, 23, 32, 33] observed the phenomenon of soliton scattering by the effect of the defect, namely, interactions between the defect and the soliton (the standing wave solution of the case $\gamma = 0$). For example, varying amplitude and velocity of the soliton, they studied how the defect is separating the soliton into two parts : one part is transmitted past the defect, the other one is captured at the defect. Holmer, Marzuola and Zworski [21, 22] gave numerical simulations and theoretical arguments on this subject. In this paper, we study the stability of the standing wave solution of (2.1) created by the Dirac delta.

A standing wave for (2.1) is a solution of the form $u(t,x) = e^{i\omega t}\varphi(x)$ where φ is required to satisfy

$$\begin{cases} -\partial_{xx}\varphi + \omega\varphi - \gamma\delta(x)\varphi - |\varphi|^{p-1}\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases} \tag{2.2}$$

Before stating our results, we introduce some notations and recall some previous results.

The space $L^r(\mathbb{R},\mathbb{C})$ will be denoted by $L^r(\mathbb{R})$ and its norm by $\|\cdot\|_r$. When $r = 2$, the space $L^2(\mathbb{R})$ will be endowed with the scalar product

$$(u,v)_2 = \mathrm{Re}\int_{\mathbb{R}} u\bar{v}dx \quad \text{for} \quad u,v \in L^2(\mathbb{R}).$$

The space $H^1(\mathbb{R},\mathbb{C})$ will be denoted by $H^1(\mathbb{R})$, its norm by $\|\cdot\|_{H^1(\mathbb{R})}$ and the duality product between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$ by $\langle \cdot,\cdot \rangle$. We write $H^1_{\mathrm{rad}}(\mathbb{R})$ for the space of radial (even) functions of $H^1(\mathbb{R})$:

$$H^1_{\mathrm{rad}}(\mathbb{R}) = \{v \in H^1(\mathbb{R});\ v(x) = v(-x), \quad x \in \mathbb{R}\}.$$

When $\gamma = 0$, the set of solutions of (2.2) has been known for a long time. In particular, modulo translation and phase, there exists a unique positive solution, which is explicitly known. This solution is even and is a ground state (see, for example, [3, 6] for such results). When $\gamma \neq 0$, an explicit solution of (2.2) was presented in [12, 18] and the following was proved in [11, 12].

Proposition 2.1. *Let $\omega > \gamma^2/4$. Then there exists a unique positive solution $\varphi_{\omega,\gamma}$ of* (2.2)*. This solution is the unique positive minimizer of*

$$d(\omega) = \begin{cases} \inf\{S_{\omega,\gamma}(v);\ v \in H^1(\mathbb{R}) \setminus \{0\},\ I_{\omega,\gamma}(v) = 0\} & \text{if} \quad \gamma \geqslant 0, \\ \inf\{S_{\omega,\gamma}(v);\ v \in H^1_{\mathrm{rad}}(\mathbb{R}) \setminus \{0\},\ I_{\omega,\gamma}(v) = 0\} & \text{if} \quad \gamma < 0, \end{cases}$$

where $S_{\omega,\gamma}$ and $I_{\omega,\gamma}$ are defined for $v \in H^1(\mathbb{R})$ by

$$\begin{aligned} S_{\omega,\gamma}(v) &= \frac{1}{2}\|\partial_x v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \\ I_{\omega,\gamma}(v) &= \|\partial_x v\|_2^2 + \omega\|v\|_2^2 - \gamma|v(0)|^2 - \|v\|_{p+1}^{p+1}. \end{aligned}$$

Furthermore, we have an explicit formula for $\varphi_{\omega,\gamma}$

$$\varphi_{\omega,\gamma}(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}. \tag{2.3}$$

The dependence of $\varphi_{\omega,\gamma}$ on ω and γ can be seen in Figure 2.1. The parameter ω affects the width and height of $\varphi_{\omega,\gamma}$: the larger ω is, the narrower and higher $\varphi_{\omega,\gamma}$ becomes, and vice versa. The sign of γ determines the profile of $\varphi_{\omega,\gamma}$ near $x = 0$. It has a "$\vee$" shape when $\gamma < 0$, and a "$\wedge$" shape when $\gamma > 0$.

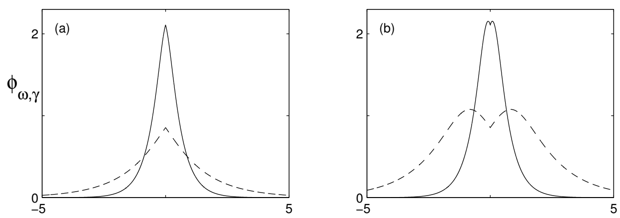

Figure 2.1 - $\varphi_{\omega,\gamma}$ as a function of x for $\omega = 4$ (solid line) and $\omega = 0.5$ (dashed line). (a) $\gamma = 1$; (b) $\gamma = -1$. Here, $p = 4$.

Remark 2.2. (i) As it was stated in [11, Remark 8 and Lemma 26], the set of solutions of (2.2)

$$\{v \in H^1(\mathbb{R}) \setminus \{0\} \text{ such that } -\partial_{xx}v + \omega v - \gamma v\delta - |v|^{p-1}v = 0\}$$

is explicitely given by $\{e^{i\theta}\varphi_{\omega,\gamma} | \, \theta \in \mathbb{R}\}$.

(ii) There is no nontrivial solution in $H^1(\mathbb{R})$ for $\omega \leqslant \gamma^2/4$.

The local well-posedness of the Cauchy problem for (2.1) is ensured by [6, Theorem 4.6.1]. Indeed, the operator $-\partial_{xx} - \gamma\delta$ is a self-adjoint operator on $L^2(\mathbb{R})$ (see [1, Chapter I.3.1] and Section 2.2 for details). Precisely, we have

Proposition 2.3. *For any $u_0 \in H^1(\mathbb{R})$, there exist $T_{u_0} > 0$ and a unique solution $u \in \mathcal{C}([0, T_{u_0}), H^1(\mathbb{R})) \cap \mathcal{C}^1([0, T_{u_0}), H^{-1}(\mathbb{R}))$ of (2.1) such that $\lim_{t \uparrow T_{u_0}} \|\partial_x u\|_2 = +\infty$ if $T_{u_0} < +\infty$. Furthermore, the conservation of energy and charge holds, that is, for any $t \in [0, T_{u_0})$ we have*

$$E(u(t)) = E(u_0), \tag{2.4}$$
$$\|u(t)\|_2^2 = \|u_0\|_2^2, \tag{2.5}$$

where the energy E is defined by

$$E(v) = \frac{1}{2}\|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \quad \textit{for} \quad v \in H^1(\mathbb{R}).$$

(see also a verification of this proposition in [12, Proposition 1]).

Remark 2.4. From the uniqueness result of Proposition 2.3 it follows that if an initial data u_0 belongs to $H^1_{\mathrm{rad}}(\mathbb{R})$ then $u(t)$ also belongs to $H^1_{\mathrm{rad}}(\mathbb{R})$ for all $t \in [0, T_{u_0})$.

We consider the stability in the following sense.

Definition 2.5. *Let φ be a solution of* (2.2). *We say that the standing wave $u(x,t) = e^{i\omega t}\varphi(x)$ is* (orbitally) stable *in $H^1(\mathbb{R})$ (resp. $H^1_{\mathrm{rad}}(\mathbb{R})$) if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property : if $u_0 \in H^1(\mathbb{R})$ (resp. $H^1_{\mathrm{rad}}(\mathbb{R})$) satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R})} < \eta$, then the solution $u(t)$ of* (2.1) *with $u(0) = u_0$ exists for any $t \geqslant 0$ and*

$$\sup_{t \in [0,+\infty)} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\varphi\|_{H^1(\mathbb{R})} < \varepsilon.$$

Otherwise, the standing wave $u(x,t) = e^{i\omega t}\varphi(x)$ is said to be (orbitally) unstable *in $H^1(\mathbb{R})$ (resp. $H^1_{\mathrm{rad}}(\mathbb{R})$).*

Remark 2.6. With this definition and Remark 2.4, it is clear that stability in $H^1(\mathbb{R})$ implies stability in $H^1_{\text{rad}}(\mathbb{R})$ and conversely that instability in $H^1_{\text{rad}}(\mathbb{R})$ implies instability in $H^1(\mathbb{R})$.

When $\gamma = 0$, the orbital stability for (2.1) has been extensively studied (see [2, 6, 7, 36, 37] and the references therein). In particular, from [7] we know that $e^{i\omega t}\varphi_{\omega,0}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega > 0$ if $1 < p < 5$. On the other hand, it was shown that $e^{i\omega t}\varphi_{\omega,0}(x)$ is unstable in $H^1(\mathbb{R})$ for any $\omega > 0$ if $p \geqslant 5$ (see [2] for $p > 5$ and [37] for $p = 5$).

In [18], Goodman, Holmes and Weinstein focused on the special case $p = 3$, $\gamma > 0$ and proved that the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is orbitally stable in $H^1(\mathbb{R})$. When $\gamma > 0$, the orbital stability and instability were completely studied in [12] : the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega > \gamma^2/4$ if $1 < p \leqslant 5$, and if $p > 5$, there exists a critical frequency $\omega_1 > \gamma^2/4$ such that $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \omega_1)$ and unstable in $H^1(\mathbb{R})$ for any $\omega > \omega_1$.

When $\gamma < 0$, Fukuizumi and Jeanjean showed the following result in [11].

Proposition 2.7. *Let $\gamma < 0$ and $\omega > \gamma^2/4$.*

(i) If $1 < p \leqslant 3$ the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1_{\text{rad}}(\mathbb{R})$.

(ii) If $3 < p < 5$, there exists $\omega_2 > \gamma^2/4$ such that the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1_{\text{rad}}(\mathbb{R})$ when $\omega > \omega_2$ and unstable in $H^1(\mathbb{R})$ when $\gamma^2/4 < \omega < \omega_2$.

(iii) If $p \geqslant 5$, then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is unstable in $H^1(\mathbb{R})$.

The critical frequency ω_2 is given by

$$\frac{J(\omega_2)(p-5)}{p-1} = \frac{\gamma}{2\sqrt{\omega_2}}\left(1 - \frac{\gamma^2}{4\omega_2}\right)^{-(p-3)/(p-1)},$$
$$J(\omega_2) = \int_{A(\omega_2,\gamma)}^{+\infty} \operatorname{sech}^{4/(p-1)}(y)dy, \quad A(\omega_2,\gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_2}}\right).$$

The results of stability of [11] recalled in Proposition 2.7 assert only on stability under radial perturbations. Furthermore, the nature of instability is not revealed. In this paper, we prove that there is instability in the whole space when stability holds under radial perturbation (see Theorem 2.1), and that, when $p \geqslant 5$, the instability established in [11] is strong instability (see Definition 2.9 and Theorem 2.2).

Our first main result is the following.

Theorem 2.1. *Let $\gamma < 0$ and $\omega > \gamma^2/4$.*

(i) If $1 < p \leqslant 3$ the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is unstable in $H^1(\mathbb{R})$.

(ii) If $3 < p < 5$, the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is unstable in $H^1(\mathbb{R})$ for any $\omega > \omega_2$, where ω_2 is defined in Proposition 2.7.

As in [11, 12], our stability analysis relies on the abstract theory by Grillakis, Shatah and Strauss [19, 20] for a Hamiltonian system which is invariant under a one-parameter group of operators. In trying to follow this approach the main point is to check the following two conditions.

1. *The slope condition.* The sign of $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$.

2. The *spectral condition.* The number of negative eigenvalues of the linearized operator
$$L^{\gamma}_{1,\omega} v = -\partial_{xx} v + \omega v - \gamma \delta v - p\varphi^{p-1}_{\omega,\gamma} v.$$

We refer the reader to Section 2.2 for the precise criterion and a detailed explanation on how $L^{\gamma}_{1,\omega}$ appears in the stability analysis. Making use of the explicit form (2.3) for $\varphi_{\omega,\gamma}$, the sign of $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$ was explicitly computed in [11, 12].

In [11], a spectral analysis is performed to count the number of negative eigenvalues, and it is proved that the number of negative eigenvalues of $L^{\gamma}_{1,\omega}$ in $H^1_{\text{rad}}(\mathbb{R})$ is one. This spectral analysis of $L^{\gamma}_{1,\omega}$ is relying on the variational characterization of $\varphi_{\omega,\gamma}$. However, since $\varphi_{\omega,\gamma}$ is a minimizer only in the space of radial (even) functions $H^1_{\text{rad}}(\mathbb{R})$, the result on the spectrum holds only in $H^1_{\text{rad}}(\mathbb{R})$, namely for even eigenfunctions. Therefore the number of negatives eigenvalues is known only for $L^{\gamma}_{1,\omega}$ considered in $H^1_{\text{rad}}(\mathbb{R})$. With this approach, it is not possible to see whether other negative eigenvalues appear when the problem is considered on the whole space $H^1(\mathbb{R})$.

To overcome this difficulty, we develop a perturbation method. In the case $\gamma = 0$, the spectrum of $L^{0}_{1,\omega}$ is well known by the work of Weinstein [38] (see Lemma 2.21) : there is only one negative eigenvalue, and 0 is a simple isolated eigenvalue (to see that, one proves that the kernel of $L^{0}_{1,\omega}$ is spanned by $\partial_x\varphi_{\omega,0}$, that $\partial_x\varphi_{\omega,0}$ has only one zero, and apply the Sturm Oscillation Theorem). When γ is small, $L^{\gamma}_{1,\omega}$ can be considered as a holomorphic perturbation of $L^{0}_{1,\omega}$. Using the theory of holomorphic perturbations for linear operators, we prove that the spectrum of $L^{\gamma}_{1,\omega}$ depends holomorphically on γ (see Lemma 2.22). Then the use of Taylor expansion for the second eigenvalue of $L^{\gamma}_{1,\omega}$ allows us to get the sign of the second eigenvalue when γ is small (see Lemma 2.23). A continuity argument combined with the fact that if $\gamma \neq 0$ the nullspace of $L^{\gamma}_{1,\omega}$ is zero extends the result to all $\gamma \in \mathbb{R}$ (see the proof of Lemma 2.18). See subsection 2.2.3 for details. We will see that there are two negative eigenvalues of $L^{\gamma}_{1,\omega}$ in $H^1(\mathbb{R})$ if $\gamma < 0$.

Remark 2.8. (i) Our method can be applied as well in $H^1(\mathbb{R})$ or in $H^1_{\text{rad}}(\mathbb{R})$, and for γ negative or positive (see subsections 2.2.4 and 2.2.5). Thus we can give another proof of the result of [12] in the case $\gamma > 0$ and of Proposition 2.7.

(ii) The study of the spectrum of linearized operators is often a central point when one wants to use the abstract theory of [19, 20]. See [9, 13, 14, 15, 24] among many others for related results.

The results of instability given in Theorem 2.1 and Proposition 2.7 say only that a certain solution which starts close to $\varphi_{\omega,\gamma}$ will exit from a tubular neighborhood of the orbit of the standing wave in finite time. However, as this might be of importance for the applications, we want to understand further the nature of instability. For that, we recall the concept of strong instability.

Definition 2.9. *A standing wave $e^{i\omega t}\varphi(x)$ of (2.1) is said to be* strongly unstable in $H^1(\mathbb{R})$ *if for any $\varepsilon > 0$ there exist $u_\varepsilon \in H^1(\mathbb{R})$ with $\|u_\varepsilon - \varphi\|_{H^1(\mathbb{R})} < \varepsilon$ and $T_{u_\varepsilon} < +\infty$ such that $\lim_{t \uparrow T_{u_\varepsilon}} \|\partial_x u(t)\|_2 = +\infty$, where $u(t)$ is the solution of (2.1) with $u(0) = u_\varepsilon$.*

Our second main result is the following.

Theorem 2.2. *Let $\gamma \leqslant 0$, $\omega > \gamma^2/4$ and $p \geqslant 5$. Then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is strongly unstable in $H^1(\mathbb{R})$.*

Whether the perturbed standing wave blows up or not depends on the perturbation. Indeed, in Remark 2.30 we define an invariant set of solutions and show that if we consider an initial data in this set, then the solution exists globally even when the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is strongly unstable.

We also point out that when $1 < p < 5$, it is easy to prove using the conservation laws and Gagliardo-Nirenberg inequality that the Cauchy problem in $H^1(\mathbb{R})$ associated with (2.1) is globally well posed. Accordingly, even if the standing wave may be unstable when $1 < p < 5$ (see Theorem 2.1), a strong instability cannot occur.

As in [2, 37], which deal with the classical case $\gamma = 0$, we use the virial identity for the proof of Theorem 2.2. However, even if the formal calculations are similar to those of the case $\gamma = 0$, a rigorous proof of the virial theorem does not immediately follow from the approximation by regular solutions (e.g. see [6, Proposition 6.4.2], or [16]). Indeed, the argument in [6] relies on the $H^2(\mathbb{R})$–regularity of the solutions of (2.1). Because of the defect term, we do not know if this $H^2(\mathbb{R})$–regularity still holds when $\gamma \neq 0$. Thus we need another approach. We approximate the solutions of (2.1) by solutions of the same equation where the defect is approximated by a Gaussian potential for which it is easy to have the virial theorem. Then we pass to the limit in the virial identity to obtain :

Proposition 2.10. *Let $u_0 \in H^1(\mathbb{R})$ such that $xu_0 \in L^2(\mathbb{R})$ and $u(t)$ be the solution of* (2.1) *with $u(0) = u_0$. Then the function $f : t \mapsto \|xu(t)\|_2^2$ is $\mathcal{C}^2$ and*

$$\partial_t f(t) = 4\mathrm{Im} \int_{\mathbb{R}} \bar{u} x \partial_x u dx, \tag{2.6}$$

$$\partial_{tt} f(t) = 8Q_\gamma(u(t)), \tag{2.7}$$

where Q_γ is defined for $v \in H^1(\mathbb{R})$ by

$$Q_\gamma(v) = \|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Even if we benefit from the virial identity, the proofs given in [2, 37] for the case $\gamma = 0$ do not apply to the case $\gamma < 0$. For example, the method of Weinstein [37] in the case $p = 5$ requires in a crucial way an equality between $2E$ and Q which does not hold anymore when $\gamma < 0$. Moreover, the heart of the proof of [2] consists in minimizing the functional $S_{\omega,\gamma}$ on the constraint $Q_\gamma(v) = 0$, but the standard variational methods to prove such results are not so easily applied to the case $\gamma \neq 0$. To get over these difficulties we introduce an approach based on a minimization problem involving two constraints. Using this minimization problem, we identify some invariant properties under the flow of (2.1). The combination of these invariant properties with the conservation of energy and charge allows us to prove strong instability. We mention that related techniques have been introduced in [26, 27, 28, 30, 39].

Remark 2.11. The case $\gamma < 0$, $\omega = \omega_2$ and $3 < p < 5$ cannot be treated with our approach and is left open (see Remark 2.15). In light of Theorem 2.1, we believe that the standing wave is unstable in this case, at least in $H^1(\mathbb{R})$ (see also [11, Remark 12]). When $\gamma > 0$, the case $\omega = \omega_1$ and $p > 5$ is also open (see [12, Remark 1.5]).

Let us summarize the previously known and our new rigorous results on stability in (2.1).

(i) For both positive and negative γ, there is always only one negative eigenvalue of the linearized operator in $H^1_{\text{rad}}(\mathbb{R})$ ([11], subsection 2.5). Hence, the standing wave is stable in $H^1_{\text{rad}}(\mathbb{R})$ if the slope is positive, and unstable if the slope is negative.

(ii) $\gamma > 0$. In this case the number of the negative eigenvalues of linearized operator is always one in $H^1(\mathbb{R})$. Stability is determined by the slope condition, and the standing wave is stable in $H^1_{\text{rad}}(\mathbb{R})$ if and only if it is stable in $H^1(\mathbb{R})$. Specifically ([11, 12], subsection 2.4),

  (a) $1 < p \leq 5$: Stability in $H^1(\mathbb{R})$ for any $\omega > \gamma^2/4$.

  (b) $5 < p$: Stability in $H^1(\mathbb{R})$ for $\gamma^2/4 < \omega < \omega_1$, instability in $H^1_{\text{rad}}(\mathbb{R})$ for $\omega > \omega_1$.

(iii) $\gamma < 0$. In this case the number of negative eigenvalues is always two (Lemma 2.18) and all standing waves are unstable in $H^1(\mathbb{R})$ (Theorem 2.1 and Theorem 2.2). Stability in $H^1_{\text{rad}}(\mathbb{R})$ is determined by the slope condition and is as follows ([11]):

  (a) $1 < p \leq 3$: Stability in $H^1_{\text{rad}}(\mathbb{R})$ for any $\omega > \gamma^2/4$.

  (b) $3 < p < 5$: Stability in $H^1_{\text{rad}}(\mathbb{R})$ for $\omega > \omega_2$, instability in $H^1_{\text{rad}}(\mathbb{R})$ for $\gamma^2/4 < \omega < \omega_2$.

  (c) $5 \leq p$: Strong instability in $H^1_{\text{rad}}(\mathbb{R})$ (and in $H^1(\mathbb{R})$) for any $\gamma^2/4 < \omega$ (Theorem 2.2).

There are, however, several important questions which are still open, and which we explore using numerical simulations. Our simulations suggest the following:

(i) Although an attractive defect ($\gamma > 0$) stabilizes the standing waves in the critical case ($p = 5$), their stability is weaker than in the subcritical case, in particular for $0 < \gamma \ll 1$.

(ii) Theorem 2.2 shows that instability occurs by blow-up when $\gamma < 0$ and $p \geqslant 5$. In all other cases, however, it remains to understand the nature of instability. Our simulations suggest the following:

  (a) When $\gamma > 0$, $p > 5$, and $\omega > \omega_1$, instability can occur by blow-up.

  (b) When $\gamma < 0$, $3 < p < 5$, and $\gamma^2/4 < \omega < \omega_2$, the instability in $H^1_{\text{rad}}(\mathbb{R})$ is a *finite-width instability*, i.e., the solution initially narrows down along a curve $\phi_{\omega*(t),\gamma}$, where $\omega^*(t)$ can be defined by the relation

$$\max_x \phi_{\omega^*(t),\gamma}(x) = \max_x |u(x,t)|.$$

As the solution narrows down, $\omega^*(t)$ increases and crosses from the unstable region $\omega < \omega_2$ to the stable region $\omega > \omega_2$. Subsequently, collapse is arrested at some finite width.

(c) When $\gamma < 0$, the standing waves undergo a *drift instability*, away from the (repulsive) defect, sometimes in combination with finite-width or blowup instability. Specifically,

(c.i) When $1 < p \leq 3$ and when $3 < p < 5$ and $\omega > \omega_2$ (i.e., when the standing waves are stable in $H^1_{\text{rad}}(\mathbb{R})$), the standing waves undergo a *drift instability*.

(c.ii) When $3 < p < 5$ and $\gamma^2/4 < \omega < \omega_2$, the instability in $H^1(\mathbb{R})$ is a combination of a drift instability and a finite-width instability.

(c.iii) When $p \geq 5$, the instability in $H^1(\mathbb{R})$ is a combination of a drift instability and a blowup instability.

(iii) Although when $p = 5$ and $\gamma > 0$, and when $p > 5$, $\gamma > 0$, and $\gamma^2/4 < \omega < \omega_1$ the standing wave is stable, it can collapse under a sufficiently large perturbation.

We note that all of the above holds, more generally, for NLS equations with an inhomogeneous nonlinearity [9] and with a linear potential [34].

The paper is organized as follows. In Section 2.2, we prove Theorem 2.1 and explain how our method allows us to recover the results of [11, 12]. In Section 2.3, we establish Theorem 2.2 and in Section 2.4 we prove Proposition 2.10. Numerical results are given in Section 2.5.

Throughout the paper the letter C will denote various positive constants whose exact values may change from line to line but are not essential to the analysis of the problem.

2.2 Instability with respect to non-radial perturbations

We use the general theory of Grillakis, Shatah and Strauss [20] to prove Theorem 2.1.

First, we explain how we derive a criterion for stability or instability for our case from the theory of Grillakis, Shatah and Strauss. In our case, it is clear that *Assumption 1* and *Assumption 2* of [20] are satisfied. The last assumption, *Assumption 3*, will be check in subsection 2.2.2. We consider the bilinear form

$$S''_{\omega,\gamma}(\varphi_{\omega,\gamma}) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$$

as a linear operator $H^\gamma_\omega : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$. The spectrum of H^γ_ω is the set

$$\{\lambda \in \mathbb{C} \text{ such that } H^\gamma_\omega - \lambda I \text{ is not invertible}\},$$

where I denote the usual $H^1(\mathbb{R}) - H^{-1}(\mathbb{R})$ isomorphism, and we denote

$$n(H^\gamma_\omega) := \text{ the number of negative eigenvalues of } H^\gamma_\omega.$$

Having established the assumptions of [20], the next proposition follows from [20, Instability Theorem and Stability Theorem].

Proposition 2.12. (1) *The standing wave $e^{i\omega_0 t}\varphi_{\omega_0,\gamma}(x)$ is unstable if the integer $(n(H^\gamma_{\omega_0}) - p(d''(\omega_0))$ is odd, where*

$$p(d''(\omega_0)) = \begin{cases} 1 & \text{if} \quad \partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0 \quad \text{at} \quad \omega = \omega_0, \\ 0 & \text{if} \quad \partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0 \quad \text{at} \quad \omega = \omega_0. \end{cases}$$

(2) *The standing wave $e^{i\omega_0 t}\varphi_{\omega_0,\gamma}(x)$ is stable if $(n(H^\gamma_{\omega_0}) - p(d''(\omega_0)) = 0$.*

Let us now consider the case $\gamma < 0$. It was proved in [11] that

Lemma 2.13. *Let $\gamma < 0$ and $\omega > \gamma^2/4$. We have :*

(*i*) *If $1 < p \leqslant 3$ and $\omega > \gamma^2/4$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,*

(*ii*) *If $3 < p < 5$ and $\omega > \omega_2$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,*

(*iii*) *If $3 < p < 5$ and $\gamma^2/4 < \omega < \omega_2$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$,*

(*iv*) *If $p \geqslant 5$ and $\omega > \gamma^2/4$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$.*

Thus Theorem 2.1 follows from Proposition 2.12, Lemma 2.13 and

Lemma 2.14. *If $\gamma < 0$, then $n(H^\gamma_\omega) = 2$.*

Remark 2.15. 1. Let $\gamma < 0$. In the cases $3 < p < 5$ and $\omega < \omega_2$ or $p \geq 5$ it was proved in [11] that $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$. From Lemma 2.14, we know that the number of negative eigenvalues of H^γ_ω is $n(H^\gamma_\omega) = 2$ when H^γ_ω is considered on the whole space $H^1(\mathbb{R})$. Therefore $n(H^\gamma_\omega) - p(d''(\omega)) = 2$ and this correspond to a case where the theory of [20] does not apply. However, if we consider H^γ_ω in $H^1_{\mathrm{rad}}(\mathbb{R})$, then it follows from [11] that $n(H^\gamma_\omega) = 1$, thus $n(H^\gamma_\omega) - p(d''(\omega)) = 1$. Then, Proposition 2.12 applies and allows to conclude to instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ (as it was done in [11]). But, with Remark 2.6, we can conclude that instability holds on the whole space $H^1(\mathbb{R})$. This shows that, sometimes, to introduce artificially a symmetry can be useful when one faces a case left open in [20].

2. Note that the case $\omega = \omega_2$ corresponds to $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 = 0$ $(3 < p < 5)$ and will not be treated here. In view of Theorem 2.1, we believe that the standing wave is unstable in this case, at least in $H^1(\mathbb{R})$.

We divide the rest of this section into five parts. In subsection 2.2.1 we introduce the general setting to perform our proof. In subsection 2.2.2, we study the spectrum of H_ω^γ and prove that *Assumption 3* of [20] is satisfied. Lemma 2.14 will be proved in subsection 2.2.3. Finally, we discuss the positive case and the radial case in subsections 2.2.4 and 2.2.5.

2.2.1 Setting for the spectral problem

To express H_ω^γ, it is convenient to split u in real and imaginary part : for $u \in H^1(\mathbb{R}, \mathbb{C})$ we write $u = u_1 + iu_2$ where $u_1 = \mathrm{Re}(u) \in H^1(\mathbb{R}, \mathbb{R})$ and $u_2 = \mathrm{Im}(u) \in H^1(\mathbb{R}, \mathbb{R})$. Now we set

$$H_\omega^\gamma u = L_{1,\omega}^\gamma u_1 + iL_{2,\omega}^\gamma u_2$$

where the operators $L_{1,\omega}^\gamma, L_{2,\omega}^\gamma : H^1(\mathbb{R}, \mathbb{R}) \to H^{-1}(\mathbb{R})$ are defined for $v \in H^1(\mathbb{R})$ by

$$\begin{array}{rcl} L_{1,\omega}^\gamma v & = & -\partial_{xx} v + \omega v - \gamma v\delta - p\varphi_{\omega,\gamma}^{p-1} v, \\ L_{2,\omega}^\gamma v & = & -\partial_{xx} v + \omega v - \gamma v\delta - \varphi_{\omega,\gamma}^{p-1} v. \end{array}$$

When we will work with $L_{1,\omega}^\gamma, L_{2,\omega}^\gamma$, the functions considered will be understood to be real valued.

For the spectral study of H_ω^γ, it is convenient to view H_ω^γ as an unbounded operator on $L^2(\mathbb{R})$, thus we rewrite our spectral problem in this setting. First, we redefine the two operators $L_{1,\omega}^\gamma$ and $L_{2,\omega}^\gamma$ as unbounded operators on $L^2(\mathbb{R})$. We begin by considering the bilinear forms on $H^1(\mathbb{R})$ associated with $L_{1,\omega}^\gamma$ and $L_{2,\omega}^\gamma$ by setting for $v, w \in H^1(\mathbb{R})$

$$B_{1,\omega}^\gamma(v, w) := \left\langle L_{1,\omega}^\gamma v, w \right\rangle \text{ and } B_{2,\omega}^\gamma(v, w) := \left\langle L_{2,\omega}^\gamma v, w \right\rangle,$$

which are explicitly given by

$$\begin{array}{rcl} B_{1,\omega}^\gamma(v, w) & = & \int_{\mathbb{R}} \partial_x v \partial_x w dx + \omega \int_{\mathbb{R}} vw dx - \gamma v(0)w(0) - \int_{\mathbb{R}} p\varphi_{\omega,\gamma}^{p-1} vw dx, \\ B_{2,\omega}^\gamma(v, w) & = & \int_{\mathbb{R}} \partial_x v \partial_x w dx + \omega \int_{\mathbb{R}} vw dx - \gamma v(0)w(0) - \int_{\mathbb{R}} \varphi_{\omega,\gamma}^{p-1} vw dx. \end{array} \tag{2.8}$$

Let us now consider $B_{1,\omega}^\gamma$ and $B_{2,\omega}^\gamma$ as bilinear forms on $L^2(\mathbb{R})$ with domain $D(B_{1,\omega}^\gamma) = D(B_{2,\omega}^\gamma) := H^1(\mathbb{R})$. It is clear that theses forms are bounded from below and closed. Then the theory of representation of forms by operators (see [25, VI.§2.1]) implies that we define two self-adjoint operators $\widetilde{L^{\gamma}_{1,\omega}} : D(\widetilde{L^{\gamma}_{1,\omega}}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\widetilde{L^{\gamma}_{2,\omega}} : D(\widetilde{L^{\gamma}_{2,\omega}}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by setting

$$\begin{array}{rcl} D(\widetilde{L^{\gamma}_{1,\omega}}) & := & \{v \in H^1(\mathbb{R}) | \exists w \in L^2(\mathbb{R}) \text{ s.t. } \forall z \in H^1(\mathbb{R}),\ B^{\gamma}_{1,\omega}(v,z) = (w,z)_2\}, \\ D(\widetilde{L^{\gamma}_{2,\omega}}) & := & \{v \in H^1(\mathbb{R}) | \exists w \in L^2(\mathbb{R}) \text{ s.t. } \forall z \in H^1(\mathbb{R}),\ B^{\gamma}_{2,\omega}(v,z) = (w,z)_2\}. \end{array}$$

and setting for $v \in D(\widetilde{L^{\gamma}_{1,\omega}})$ (resp. $v \in D(\widetilde{L^{\gamma}_{2,\omega}})$) that $\widetilde{L^{\gamma}_{1,\omega}} v := w$ (resp. $\widetilde{L^{\gamma}_{2,\omega}} v := w$), where w is the (unique) function of $L^2(\mathbb{R})$ which satisfies $B^{\gamma}_{1,\omega}(v,z) = (w,z)_2$ (resp. $B^{\gamma}_{2,\omega}(v,z) = (w,z)_2$) for all $z \in H^1(\mathbb{R})$.

For notational simplicity, we drop the tilde over $\widetilde{L^{\gamma}_{1,\omega}}$ and $\widetilde{L^{\gamma}_{2,\omega}}$.

It turns out that we are able to describe explicitly $L^{\gamma}_{1,\omega}$ and $L^{\gamma}_{2,\omega}$.

Lemma 2.16. *The domain of $L^{\gamma}_{1,\omega}$ and of $L^{\gamma}_{2,\omega}$ in $L^2(\mathbb{R})$ is*

$$D_\gamma = \{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\});\ \partial_x v(0^+) - \partial_x v(0^-) = -\gamma v(0)\}$$

and for $v \in D_\gamma$ the operators are given by

$$\begin{array}{rcl} L^{\gamma}_{1,\omega} v & = & -\partial_{xx} v + \omega v - p\varphi^{p-1}_{\omega,\gamma} v, \\ L^{\gamma}_{2,\omega} v & = & -\partial_{xx} v + \omega v - \varphi^{p-1}_{\omega,\gamma} v. \end{array} \tag{2.9}$$

Proof. The proof for $L^{\gamma}_{2,\omega}$ being similar to the one of $L^{\gamma}_{1,\omega}$ we only deal with $L^{\gamma}_{1,\omega}$. The form $B^{\gamma}_{1,\omega}$ can be decomposed into $B^{\gamma}_{1,\omega} = B^{\gamma}_{1,1} + B^{\gamma}_{1,2,\omega}$ with $B^{\gamma}_{1,1} : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ and $B^{\gamma}_{1,2,\omega} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\begin{array}{rcl} B^{\gamma}_{1,1}(v,z) & = & \int_{\mathbb{R}} \partial_x v \partial_x z dx - \gamma v(0) z(0), \\ B^{\gamma}_{1,2,\omega}(v,z) & = & \omega \int_{\mathbb{R}} vz dx - \int_{\mathbb{R}} p\varphi^{p-1}_{\omega,\gamma} vz dx. \end{array} \tag{2.10}$$

If we denote by T_1 (resp. T_2) the self-adjoint operator on $L^2(\mathbb{R})$ associated with $B^{\gamma}_{1,1}$ (resp $B^{\gamma}_{1,2,\omega}$), it is clear that $D(T_2) = L^2(\mathbb{R})$ and

$$D(L^{\gamma}_{1,\omega}) = D(T_1).$$

Let $v \in D_\gamma$ and $w \in L^2(\mathbb{R})$ be such that $B^{\gamma}_{1,1}(v,z) = (w,z)_2$ for any $z \in H^1(\mathbb{R})$. If $z \in H^1(\mathbb{R})$ is such that $z(0) = 0$, we have

$$B^{\gamma}_{1,1}(v,z) = \int_{\mathbb{R}} \partial_x v \partial_x z dx = \int_{\mathbb{R}} wz dx,$$

therefore $v \in H^2(\mathbb{R} \setminus \{0\})$ and $-\partial_{xx} v = w$. Let $z \in H^1(\mathbb{R})$ be such that $z(0) \neq 0$. On one hand, we have

$$B_{1,1}^{\gamma}(v,z) = \int_{\mathbb{R}} \partial_x v \partial_x z dx - \gamma v(0) z(0).$$

And on other hand

$$\begin{aligned} B_{1,1}^{\gamma}(v,z) &= (w,z)_2, \\ &= \int_{-\infty}^{0} (-\partial_{xx} v) z dx + \int_{0}^{+\infty} (-\partial_{xx} v) z dx, \\ &= -z(0)\partial_x v(0-) + \int_{-\infty}^{0} \partial_x v \partial_x z dx + z(0)\partial_x v(0+) + \int_{0}^{+\infty} \partial_x v \partial_x z dx, \\ &= \int_{\mathbb{R}} \partial_x v \partial_x z dx + z(0)(\partial_x v(0+) - \partial_x v(0-)). \end{aligned}$$

Therefore

$$\partial_x v(0^+) - \partial_x v(0^-) = -\gamma v(0),$$

which ends the proof. □

2.2.2 Verification of *Assumption 3*

To check [20, *Assumption 3*] is equivalent to check that the following lemma holds.

Lemma 2.17. *Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega > \gamma^2/4$.*

(i) *The operator H_ω^γ has only a finite number of negative eigenvalues,*

(ii) *The kernel of H_ω^γ is* span$\{i\varphi_{\omega,\gamma}\}$,

(iii) *The rest of the spectrum of H_ω^γ is positive and bounded away from* 0.

Our proof of Lemma 2.17 borrows some elements of [11]. In particular, (ii) in Lemma 2.17 corresponds to [11, Lemma 28 and Lemma 31].

Proof of Lemma 2.17. We start by showing that (i) and (iii) are satisfied. We work on $L_{1,\omega}^\gamma$ and $L_{2,\omega}^\gamma$. The essential spectrum of T_1 (see the proof of Lemma 2.16) is $\sigma_{\text{ess}}(T_1) = [0,+\infty)$. This is standard when $\gamma = 0$ and a proof for $\gamma \neq 0$ can be found in [1, Theorem I-3.1.4]. From Weyl's theorem (see [25, Theorem IV-5.35]), the essential spectrum of both operators $L_{1,\omega}^\gamma$ and $L_{2,\omega}^\gamma$ is $[\omega,+\infty)$. Since both operators are bounded from below, there can be only finitely many isolated eigenvalues (of finite multiplicity) in $(-\infty,\omega')$ for any $\omega' < \omega$. Then (i) and (iii) follow easily.

Next, we consider (ii). Since $\varphi_{\omega,\gamma}$ satisfies $L^{\gamma}_{2,\omega}\varphi_{\omega,\gamma} = 0$ and $\varphi_{\omega,\gamma} > 0$, the first eigenvalue of $L^{\gamma}_{2,\omega}$ is 0 and the rest of the spectrum is positive . This is classical for $\gamma = 0$ and can be easily proved for $\gamma \neq 0$, see [4, Chapter 2, Section 2.3, Paragraph 3]. Thus to ensure that the kernel of H^{γ}_{ω} is reduced to $\operatorname{span}\{i\varphi_{\omega,\gamma}\}$ it is enough to prove that the kernel of $L^{\gamma}_{1,\omega}$ is $\{0\}$. It is equivalent to prove that 0 is the unique solution of

$$L^{\gamma}_{1,\omega}u = 0,\ u \in D(L^{\gamma}_{1,\omega}). \tag{2.11}$$

To be more precise, the solutions of (2.11) satisfy

$$u \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}), \tag{2.12}$$

$$-\partial_{xx}u + \omega u - p\varphi^{p-1}_{\omega,\gamma}u = 0, \tag{2.13}$$

$$\partial_x u(0+) - \partial_x u(0-) = -\gamma u(0). \tag{2.14}$$

Consider first (2.13) on $(0, +\infty)$. If we look at (2.2) only on $(0, +\infty)$, we see that $\varphi_{\omega,\gamma}$ satisfies

$$-\partial_{xx}\varphi_{\omega,\gamma} + \omega\varphi_{\omega,\gamma} - \varphi^{p}_{\omega,\gamma} = 0 \text{ on } (0, +\infty). \tag{2.15}$$

If we differentiate (2.15) with respect to x (which is possible because $\varphi_{\omega,\gamma}$ is smooth on $(0, +\infty)$), we see that $\partial_x\varphi_{\omega,\gamma}$ satisfies (2.13) on $(0, +\infty)$. Since we look for solutions in $L^2(\mathbb{R})$ (in fact solutions going to 0 at infinity), it is standard that every solution of (2.13) in $(0, +\infty)$ is of the form $\mu\partial_x\varphi_{\omega,\gamma}$, $\mu \in \mathbb{R}$ (see, for example, [4, Chapter 2, Theorem 3.3]). A similar argument can be applied to (2.13) on $(-\infty, 0)$, thus every solution of (2.13) in $(-\infty, 0)$ is of the form $\nu\partial_x\varphi_{\omega,\gamma}$, $\nu \in \mathbb{R}$.

Now, let u be a solution of (2.12)-(2.14). Then there exists $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}$ such that

$$\begin{aligned} u &= \nu\partial_x\varphi_{\omega,\gamma} \quad \text{on} \quad (-\infty, 0),\\ u &= \mu\partial_x\varphi_{\omega,\gamma} \quad \text{on} \quad (0, +\infty). \end{aligned}$$

Since $u \in H^1(\mathbb{R})$, u is continuous at 0, thus we must have $\mu = -\nu$, that is, u is of the form

$$\begin{aligned} &u = -\mu\partial_x\varphi_{\omega,\gamma} \text{ on } (-\infty, 0),\\ &u = \quad \mu\partial_x\varphi_{\omega,\gamma} \text{ on } (0, +\infty),\\ &u(0) = -\mu\partial_x\varphi_{\omega,\gamma}(0-) = \mu\partial_x\varphi_{\omega,\gamma}(0+) = \frac{-\mu}{2}\gamma\varphi_{\omega,\gamma}(0). \end{aligned}$$

Furthermore, u should satisfies the jump condition (2.14). Since $\varphi_{\omega,\gamma}$ satisfies

$$\partial_{xx}\varphi_{\omega,\gamma}(0-) = \partial_{xx}\varphi_{\omega,\gamma}(0+) = \omega\varphi_{\omega,\gamma}(0) - \varphi^{p}_{\omega,\gamma}(0),$$

if we suppose $\mu \neq 0$ then (2.14) reduces to

$$\varphi^{p-1}_{\omega,\gamma}(0) = \frac{4\omega - \gamma^2}{4}.$$

But from (2.3) we know that

$$\varphi_{\omega,\gamma}^{p-1}(0) = \frac{p+1}{8}(4\omega - \gamma^2).$$

It is a contradiction, therefore $\mu = 0$. In conclusion, $u \equiv 0$ on $\mathbb{R}$, and the lemma is proved. □

2.2.3 Count of the number of negative eigenvalues

In this subsection, we prove Lemma 2.14. First, we remark that, as it was shown in the proof of Lemma 2.17, 0 is the first eigenvalue of $L_{2,\omega}^{\gamma}$. Thus $n(H_{\omega}^{\gamma}) = n(L_{1,\omega}^{\gamma})$, where $n(L_{1,\omega}^{\gamma})$ is the number of negative eigenvalues of $L_{1,\omega}^{\gamma}$. Therefore, Lemma 2.14 follows from

Lemma 2.18. *Let $\gamma < 0$ and $\omega > \gamma^2/4$. Then $n(L_{1,\omega}^{\gamma}) = 2$.*

Our proof of Lemma 2.18 is divided into two steps. First, we use a perturbative approach to prove that, if γ is close to 0 and negative, $L_{1,\omega}^{\gamma}$ has two negative eigenvalues (Lemma 2.23). To do this, we have to ensure that the eigenvalues and the eigenvectors are regular enough with respect to γ (Lemma 2.22) to make use of Taylor formula. It follows from the use of the analytic perturbation theory of operators (see [25, 31]). The second step consists in extending the result of the first step to any values of $\gamma < 0$. Our argument relies on the continuity of the spectral projections with respect to γ and it is crucial, as it was proved in Lemma 2.17, that 0 can not be an eigenvalue of $L_{1,\omega}^{\gamma}$ (see [13, 14] for related arguments).

We fix $\omega > \gamma^2/4$. For the sake of simplicity we denote $L_{1,\omega}^{\gamma}$ by L_1^{γ} and $\varphi_{\omega,\gamma}$ by φ_{γ}, and so on in this section.

Lemma 2.19. *As a function of γ, (L_1^{γ}) is a real-holomorphic family of self-adjoint operators (of type (B) in the sense of Kato).*

Proof. We recall that L_1^{γ} is defined with the help of a bilinear form B_1^{γ} (see (2.8)). To prove the holomorphicity of (L_1^{γ}) it is enough to prove that (B_1^{γ}) is bounded from below and closed, and that for any $v \in H^1(\mathbb{R})$ the function $B_1^{\gamma}(v) : \gamma \mapsto B_1^{\gamma}(v, v)$ is holomorphic (see [25, Theorem VII-4.2]). It is clear that B_1^{γ} is bounded from below and closed on the same domain $H^1(\mathbb{R})$ for all γ, thus we just have to check the holomorphicity of $B_1^{\gamma}(v) : \gamma \mapsto B_1^{\gamma}(v, v)$ for any $v \in H^1(\mathbb{R})$. We recall the decomposition of B_1^{γ} into $B_{1,1}^{\gamma}$ and $B_{1,2}^{\gamma}$ (see (2.10)). We see that $B_{1,1}^{\gamma}(v)$ is clearly holomorphic in γ. From the explicit form of φ_{γ} (see (2.3)) it is clear that $\gamma \mapsto \varphi_{\gamma}^{p-1}(x)$ is holomorphic in γ for any $x \in \mathbb{R}$. It then also follows that $\gamma \mapsto B_{1,2}^{\gamma}(v)$ is holomorphic. □

Remark 2.20. There exists another way to show that (L_1^γ) is a real-holomorphic family with respect to $\gamma \in \mathbb{R}$. We can use the explicit resolvent formula in [1],

$$(T_1 - k^2)^{-1} = (-\partial_{xx} - k^2)^{-1} + 2\gamma k(-i\gamma + 2k)^{-1}(\overline{G_k(\cdot)}, \cdot)G_k(\cdot),$$

where $k^2 \in \rho(T_1)$, $\mathrm{Im}k > 0$, $G_k(x) = (i/2k)e^{ik|x|}$, to verify the holomorphicity.

The following classical result of Weinstein [38] gives a precise description of the spectrum of the operator we want to perturb.

Lemma 2.21. *The operator L_1^0 has exactly one negative simple isolated first eigenvalue. The second eigenvalue is* 0, *and it is simple and isolated. The nullspace is* $\mathrm{span}\{\partial_x\varphi_0\}$, *and the rest of the spectrum is positive.*

Combining Lemma 2.19 and Lemma 2.21, we can apply the theory of analytic perturbations for linear operators (see [25, VII.§1.3]) to get the following lemma. Actually, the perturbed eigenvalues are holomorphic since they are simple.

Lemma 2.22. *There exist $\gamma_0 > 0$ and two functions $\lambda : (-\gamma_0, \gamma_0) \mapsto \mathbb{R}$ and $f : (-\gamma_0, \gamma_0) \mapsto L^2(\mathbb{R})$ such that*

- (i) $\lambda(0) = 0$ *and* $f(0) = \partial_x\varphi_0$,
- (ii) *For all $\gamma \in (-\gamma_0, \gamma_0)$, $\lambda(\gamma)$ is the simple isolated second eigenvalue of L_1^γ and $f(\gamma)$ is an associated eigenvector,*
- (iii) *$\lambda(\gamma)$ and $f(\gamma)$ are holomorphic in $(-\gamma_0, \gamma_0)$.*

Furthermore, $\gamma_0 > 0$ can be chosen small enough to ensure that, expect the two first eigenvalues, the spectrum of L_1^γ is positive.

Now we investigate how the perturbed second eigenvalue moves depending on the sign of γ.

Lemma 2.23. *There exists $0 < \gamma_1 < \gamma_0$ such that $\lambda(\gamma) < 0$ for any $-\gamma_1 < \gamma < 0$ and $\lambda(\gamma) > 0$ for any $0 < \gamma < \gamma_1$.*

Proof of Lemma 2.23. We develop the functions $\lambda(\gamma)$ and $f(\gamma)$ of Lemma 2.22. There exist $\lambda_0 \in \mathbb{R}$ and $f_0 \in L^2(\mathbb{R})$ such that for γ close to 0 we have

$$\lambda(\gamma) = \gamma\lambda_0 + O(\gamma^2), \tag{2.16}$$
$$f(\gamma) = \partial_x\varphi_0 + \gamma f_0 + O(\gamma^2). \tag{2.17}$$

From the explicit expression (2.3) of φ_γ, we deduce that there exists $g_0 \in H^1(\mathbb{R})$ such that for γ close to 0 we have

$$\varphi_\gamma = \varphi_0 + \gamma g_0 + O(\gamma^2). \tag{2.18}$$

Furthermore, using (2.18) to substitute into (2.2) and differentiating (2.2) with respect to γ, we obtain

$$\langle L_1^0 g_0, \psi \rangle = \varphi_0(0)\psi(0), \tag{2.19}$$

for any $\psi \in H^1(\mathbb{R})$.

To develop λ_0 with respect to γ, we compute $(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2$ in two different ways.

On one hand, using $L_1^\gamma f(\gamma) = \lambda(\gamma) f(\gamma)$, (2.16) and (2.17) leads us to

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = \lambda_0\gamma\|\partial_x\varphi_0\|_2^2 + O(\gamma^2). \tag{2.20}$$

On the other hand, since L_1^γ is self-adjoint, we get

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = (f(\gamma), L_1^\gamma\partial_x\varphi_0)_2. \tag{2.21}$$

Here we note that $\partial_x\varphi_0 \in D(L_1^\gamma)$: indeed, $\partial_x\varphi_0 \in H^2(\mathbb{R})$ and $\partial_x\varphi_0(0) = 0$. We compute the right hand side of (2.21). We use (2.9), $L_1^0\partial_x\varphi_0 = 0$, and (2.18) to obtain

$$\begin{aligned} L_1^\gamma\partial_x\varphi_0 &= p(\varphi_0^{p-1} - \varphi_\gamma^{p-1})\partial_x\varphi_0, \\ &= -\gamma p(p-1)\varphi_0^{p-2} g_0 \partial_x\varphi_0 + O(\gamma^2). \end{aligned} \tag{2.22}$$

Hence, it follows from (2.17) that

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = -(\partial_x\varphi_0, \gamma g_0 p(p-1)\varphi_0^{p-2}\partial_x\varphi_0)_2 + O(\gamma^2). \tag{2.23}$$

Now, as it was remarked in [9, Lemma 28], it is easy to see that using (2.2) with $\gamma = 0$ we get

$$L_1^0(\omega\varphi_0 - \varphi_0^{p-1}) = p(p-1)\varphi_0^{p-2}(\partial_x\varphi_0)^2, \tag{2.24}$$

which combined with (2.23) gives

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = -\gamma\left\langle L_1^0 g_0, \omega\varphi_0 - \varphi_0^p\right\rangle + O(\gamma^2). \tag{2.25}$$

Finally, with (2.19) we obtain from (2.25)

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = -\gamma(\omega\varphi_0(0)^2 - \varphi_0(0)^{p+1}) + O(\gamma^2). \tag{2.26}$$

Combining (2.26) and (2.20) we obtain

$$\lambda_0 = -\frac{\omega\varphi_0(0)^2 - \varphi_0(0)^{p+1}}{\|\partial_x\varphi_0\|_2^2} + O(\gamma).$$

It follows that λ_0 is positive for sufficiently small $|\gamma|$, which in view of (2.16) ends the proof. $\square$

We are now in position to prove Lemma 2.18.

Proof of Lemma 2.18. Let γ_∞ be defined by

$$\gamma_\infty = \inf\{\tilde{\gamma} < 0;\ L_1^\gamma \text{ has exactly two negative eigenvalues for all } \gamma \in (\tilde{\gamma}, 0)\}.$$

From Lemma 2.23, we know that γ_∞ is well defined and $\gamma_\infty \in [-\infty, 0)$. Arguing by contradiction, we suppose $\gamma_\infty > -\infty$.

Let N be the number of negative eigenvalues of $L_1^{\gamma_\infty}$. Denote the first eigenvalue of $L_1^{\gamma_\infty}$ by Λ_{γ_∞}. Let Γ be defined by

$$\Gamma = \{z \in \mathbb{C};\ z = z_1 + iz_2,\ (z_1, z_2) \in [-b, 0] \times [-a, a],\ \text{for some } a > 0, b > |\Lambda_{\gamma_\infty}|\}.$$

From Lemma 2.17, we know that $L_1^{\gamma_\infty}$ does not admit zero as eigenvalue. Thus Γ define a contour in $\mathbb{C}$ of the segment $[\Lambda_{\gamma_\infty}, 0]$ containing no positive part of the spectrum of $L_1^{\gamma_\infty}$, and without any intersection with the spectrum of $L_1^{\gamma_\infty}$. It is easily seen (for example, along the lines of the proof of [25, Theorem VII-1.7]) that there exists a small $\gamma_* > 0$ such that for any $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$, we can define a holomorphic projection on the negative part of the spectrum of L_1^γ contained in Γ by

$$\Pi(\gamma) = \frac{-1}{2\pi i}\int_\Gamma (L_1^\gamma - z)^{-1} dz.$$

Let us insist on the fact that we can choose Γ independently of the parameter γ because 0 is not an eigenvalue of L_1^γ for all γ.

Since Π is holomorphic, Π is continuous in γ, then by a classical connectedness argument (for example, see [25, Lemma I-4.10]), we know that $\dim(\mathrm{Ran}\ \Pi(\gamma)) = N$ for any $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$. Furthermore, N is exactly the number of negative eigenvalues of L_1^γ when $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$: indeed, if L_1^γ has a negative eigenvalue outside of Γ it suffice to enlarge Γ (i.e., enlarge b) until it contains this eigenvalue to raise a contradiction since then $L_1^{\gamma_\infty}$ would have, at least, $N+1$ eigenvalues. Now by the definition of γ_∞, $L_1^{\gamma_\infty+\gamma_*}$ has two negative eigenvalues and thus we see that L_1^γ has two negative eigenvalues for all $\gamma \in [\gamma_\infty - \gamma^*, 0[$ contradicting the definition of γ_∞.

Therefore $\gamma_\infty = -\infty$. $\square$

Remark 2.24. In [11, Lemma 32], the authors proved that there are *at most* two negative eigenvalues of L_1^γ in $H^1(\mathbb{R})$ using variational methods. In our present proof, we can directly show that there are exactly two negative eigenvalues.

2.2.4 The case $\gamma > 0$

The proof of Lemma 2.18 can be easily adapted to the case $\gamma > 0$, and with Lemma 2.23 we can infer that L_1^γ has only one simple negative eigenvalue when $\gamma > 0$. Since $n(H^\gamma) = n(L_1^\gamma)$, it follows that (in Lemmas 2.25, 2.26 and Proposition 2.27, there is no omission of the parameter ω)

Lemma 2.25. *Let $\gamma > 0$ and $\omega > \gamma^2/4$. Then the operator H_ω^γ has only one negative eigenvalue, that is $n(H_\omega^\gamma) = 1$.*

When $\gamma > 0$, the sign of $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$ was computed in [12]. Precisely :

Lemma 2.26. *Let $\gamma > 0$ and $\omega > \gamma^2/4$. We have :*

(i) If $1 < p \leqslant 5$ and $\omega > \gamma^2/4$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,

(ii) If $p > 5$ and $\gamma^2/4 < \omega < \omega_1$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,

(iii) If $p > 5$ and $\omega > \omega_1$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$.

Here ω_1 is defined as follows:

$$\frac{p-5}{p-1} J(\omega_1) = \frac{\gamma}{2\sqrt{\omega_1}} \left(1 - \frac{\gamma^2}{4\omega_1}\right)^{-(p-3)/(p-1)},$$
$$J(\omega_1) = \int_{A(\omega_1,\gamma)}^{\infty} \operatorname{sech}^{4/(p-1)}(y)dy, \quad A(\omega_1,\gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_1}}\right).$$

Then, using Lemma 2.25, Lemma 2.26 and Proposition 2.12, we can give an alternative proof of [12, Theorem 1] (see also [11, Remark 33]). Precisely, we obtain :

Proposition 2.27. *Let $\gamma > 0$ and $\omega > \gamma^2/4$.*

(i) Let $1 < p \leqslant 5$. Then $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, +\infty)$.

(ii) Let $p > 5$. Then $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \omega_1)$, and unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1, +\infty)$.

2.2.5 The radial case

Before we start to discuss the stability in the radial case, we mention the following remarkable fact.

Lemma 2.28. *The function $f(\gamma)$ defined in Lemma 2.22 and corresponding to the second negative eigenvalue of L_1^γ can be extended to $(-\infty, +\infty)$. Furthermore, $f(\gamma) \in H^1(\mathbb{R})$ is an odd function, for each $\gamma \in (-\infty, +\infty)$.*

Proof. First, the extension of f to $(-\infty, 0]$ is easily deduce from the proof of Lemma 2.18 and [25, VII.§3.2]. The details are left to the reader.

Secondly, as it was observed in [9, 11], the eigenvectors of L_1^γ are even or odd. Indeed, let ξ be an eigenvalue of L_1^γ with eigenvector $v \in D(L_1^\gamma)$. Then clearly $\tilde{v}$ with $\tilde{v}(x) = v(-x)$ is also an eigenvector associated to ξ. In particular, v and $\tilde{v}$ satisfy both

$$-\partial_{xx} v + (\omega - \xi) v - p\varphi_\gamma^{p-1} v = 0 \text{ on } [0, +\infty),$$

thus there exists $\eta \in \mathbb{R}$ such that $v = \eta \tilde{v}$ on $[0, +\infty)$ (this is standard, see, for example, [4, Chapter 2, Theorem 3.3]). If $v(0) \neq 0$, it is immediate that $\eta = 1$. If $v(0) = 0$, then $\partial_x v(0+) \neq 0$ (otherwise the Cauchy-Lipschitz Theorem leads to $v \equiv 0$), and it is also immediate that $\eta = -1$. Arguing in a same way on $(-\infty, 0]$, we conclude that v is even or odd, and in particular v is even if and only if $v(0) \neq 0$.

Finally, we prove the last statement only for the case $\gamma < 0$ since the case $\gamma > 0$ is similar. We remark that $\partial_x \varphi_0$ is odd. Since $\lim_{\gamma \to 0} (f(\gamma), \partial_x \varphi_0)_2 = \|\partial_x \varphi_0\|_2^2 \neq 0$, we have $(f(\gamma), \partial_x \varphi_0)_2 \neq 0$ for γ close to 0, thus $f(\gamma)$ cannot be even, and therefore $f(\gamma)$ is odd. Let $\tilde{\gamma}_\infty$ be

$$\tilde{\gamma}_\infty = \inf\{\tilde{\gamma} < 0;\ f(\gamma) \text{ is odd for any } \gamma \in (\tilde{\gamma}, 0]\}.$$

We suppose that $\tilde{\gamma}_\infty > -\infty$. If $f(\tilde{\gamma}_\infty)$ is odd, by continuity in γ of $f(\gamma)$, there exists $\varepsilon > 0$ such that $f(\tilde{\gamma}_\infty - \varepsilon)$ is odd which is a contradiction with the definition of $\tilde{\gamma}_\infty$, thus $f(\tilde{\gamma}_\infty)$ is even. Now, $f(\tilde{\gamma}_\infty)$ is the limit of odd functions, thus is odd. The only possibility to have $f(\tilde{\gamma}_\infty)$ both even and odd is $f(\tilde{\gamma}_\infty) \equiv 0$, which is impossible because $f(\tilde{\gamma}_\infty)$ is an eigenvector. $\square$

We can deduce the number of negative eigenvalues of L_1^γ in the radial case from the result on the eigenvalues of L_1^γ considered in the whole space $L^2(\mathbb{R})$. Indeed, Lemma 2.28 ensures that the second eigenvalue of L_1^γ considered in the whole space $L^2(\mathbb{R})$ is associated with an odd eigenvector, and thus disappears when the problem is restricted to the subspace of radial functions. Furthermore, since $\varphi_\gamma \in H^1_{\text{rad}}(\mathbb{R})$

and $\langle L_1^\gamma \varphi_\gamma, \varphi_\gamma \rangle < 0$, we can infer that the first negative eigenvalue of L_1^γ is still present when the problem is restricted to sets of radial functions. Recalling that $n(H^\gamma) = n(L_1^\gamma)$, we obtain.

Lemma 2.29. *Let $\gamma < 0$. Then the operator H^γ considered on $H^1_{\mathrm{rad}}(\mathbb{R})$ has only one negative eigenvalue, that is $n(H^\gamma) = 1$.*

Combining Lemma 2.29, Lemma 2.13 and Proposition 2.12, we recover the results of [11] recalled in Proposition 2.7.

Alternatively, subsection 2.2.3 can be adapted to the radial case. All the function spaces should be reduced to spaces of even functions, and Lemma 2.29 can also be proved in this way.

2.3 Strong instability

This section is devoted to the proof of Theorem 2.2.

We begin by introducing some notations

$$\mathscr{M} = \{v \in H^1_{\mathrm{rad}}(\mathbb{R}) \setminus \{0\}; Q_\gamma(v) = 0, I_{\omega,\gamma}(v) \leqslant 0\},$$

$$d_{\mathscr{M}} = \inf\{S_{\omega,\gamma}(v); v \in \mathscr{M}\},$$

where $S_{\omega,\gamma}$ and $I_{\omega,\gamma}$ are defined in Proposition 2.1 and Q_γ in Proposition 2.10.

Our proof is divided in three steps.

Step 1. We prove that $\varphi_{\omega,\gamma}$ is a minimizer of $d_{\mathscr{M}}$.

Because of Pohozaev identity $Q_\gamma(\varphi_{\omega,\gamma}) = 0$ (see [3]), it is clear that $d_{\mathscr{M}} \leqslant d(\omega)$, thus we only have to show $d_{\mathscr{M}} \geqslant d(\omega)$. Let $v \in \mathscr{M}$. If $I_{\omega,\gamma}(v) = 0$, we have $S_{\omega,\gamma}(v) \geqslant d(\omega)$, therefore we suppose $I_{\omega,\gamma}(v) < 0$. For $\alpha > 0$, let v^α be such that $v^\alpha(x) = \alpha^{1/2} v(\alpha x)$. We have

$$I_{\omega,\gamma}(v^\alpha) = \alpha^2 \|\partial_x v\|_2^2 + \omega \|v\|_2^2 - \gamma\alpha |v(0)|^2 - \alpha^{(p-1)/2} \|v\|_{p+1}^{p+1},$$

thus $\lim_{\alpha\to 0} I_{\omega,\gamma}(v^\alpha) = \omega\|v\|_2^2 > 0$, and by continuity there exists $0 < \alpha_0 < 1$ such that $I_{\omega,\gamma}(v^{\alpha_0}) = 0$. Therefore

$$S_{\omega,\gamma}(v^{\alpha_0}) \geqslant d(\omega). \tag{2.27}$$

Consider now $\dfrac{\partial}{\partial\alpha}S_{\omega,\gamma}(v^\alpha) = \alpha\|\partial_x v\|_2^2 - \dfrac{\gamma}{2}|v(0)|^2 - \dfrac{p-1}{2(p+1)}\alpha^{(p-3)/2}\|v\|_{p+1}^{p+1}$. Since $p \geqslant 5$ and $Q_\gamma(v) = 0$, we have for $\alpha \in [0,1]$

$$\frac{\partial}{\partial\alpha}S_{\omega,\gamma}(v^\alpha) \geqslant \alpha Q_\gamma(v) - \frac{\gamma}{2}(1-\alpha)|v(0)|^2 = -\frac{\gamma}{2}(1-\alpha)|v(0)|^2$$

and thus $\dfrac{\partial}{\partial\alpha}S_{\omega,\gamma}(v^\alpha) \geqslant 0$ for all $\alpha \in [0,1]$, which leads to $S_{\omega,\gamma}(v) \geqslant S_{\omega,\gamma}(v^{\alpha_0})$. It follows by (2.27) that $S_{\omega,\gamma}(v) \geqslant d(\omega)$, which concludes $d_{\mathscr{M}} = d(\omega)$.

Step 2. We construct a sequence of initial data $\varphi^\alpha_{\omega,\gamma}$ satisfying the following properties :

$$S_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma}) < d(\omega), I_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma}) < 0 \text{ and } Q_\gamma(\varphi^\alpha_{\omega,\gamma}) < 0.$$

These properties are invariant under the flow of (2.1).

For $\alpha > 0$, we define $\varphi^\alpha_{\omega,\gamma}$ by $\varphi^\alpha_{\omega,\gamma}(x) = \alpha^{1/2}\varphi_{\omega,\gamma}(\alpha x)$. Since $p \geqslant 5$, $\gamma < 0$ and $Q_\gamma(\varphi_{\omega,\gamma}) = 0$, easy computations permit to obtain

$$\frac{\partial^2}{\partial\alpha^2}S_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma})_{|\alpha=1} < 0, \frac{\partial}{\partial\alpha}I_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma})_{|\alpha=1} < 0 \text{ and } \frac{\partial}{\partial\alpha}Q_\gamma(\varphi^\alpha_{\omega,\gamma})_{|\alpha=1} < 0,$$

and thus for any $\alpha > 1$ close enough to 1 we have

$$S_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma}) < S_{\omega,\gamma}(\varphi_{\omega,\gamma}), I_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma}) < 0 \text{ and } Q_\gamma(\varphi^\alpha_{\omega,\gamma}) < 0. \tag{2.28}$$

Now fix a $\alpha > 1$ such that (2.28) is satisfied, and let $u^\alpha(t,x)$ be the solution of (2.1) with $u^\alpha(0) = \varphi^\alpha_{\omega,\gamma}$. Since $\varphi^\alpha_{\omega,\gamma}$ is radial, $u^\alpha(t)$ is also radial for all $t > 0$ (see Remark 2.4). We claim that the properties described in (2.28) are invariant under the flow of (2.1). Indeed, since from (2.4) and (2.5) we have for all $t > 0$

$$S_{\omega,\gamma}(u^\alpha(t)) = S_{\omega,\gamma}(\varphi^\alpha_{\omega,\gamma}) < S_{\omega,\gamma}(\varphi_{\omega,\gamma}), \tag{2.29}$$

we infer that $I_{\omega,\gamma}(u^\alpha(t)) \neq 0$ for any $t \geqslant 0$, and by continuity we have $I_{\omega,\gamma}(u^\alpha(t)) < 0$ for all $t \geqslant 0$. It follows that $Q_\gamma(u^\alpha(t)) \neq 0$ for any $t \geqslant 0$ (if not $u^\alpha(t) \in \mathscr{M}$ and thus $S_{\omega,\gamma}(u^\alpha(t)) \geqslant S_{\omega,\gamma}(\varphi_{\omega,\gamma})$ which contradicts (2.29)), and by continuity we have $Q_\gamma(u^\alpha(t)) < 0$ for all $t \geqslant 0$.

Step 3. We prove that $Q_\gamma(u^\alpha)$ stays negative and away from 0 for all $t \geqslant 0$.

Let $t > 0$ be arbitrary chosen, define $v = u^\alpha(t)$ and for $\beta > 0$ let v^β be such that $v^\beta(x) = v(\beta x)$. Then we have

$$Q_\gamma(v^\beta) = \beta\|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \beta^{-1}\frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1},$$

thus $\lim_{\beta\to+\infty} Q_\gamma(v^\beta) = +\infty$, and by continuity there exists β_0 such that $Q_\gamma(v^{\beta_0}) = 0$. If $I_{\omega,\gamma}(v^{\beta_0}) \leqslant 0$, we keep β_0 unchanged; otherwise, we replace it by $\tilde{\beta}_0$ such that $1 < \tilde{\beta}_0 < \beta_0$, $I_{\omega,\gamma}(v^{\tilde{\beta}_0}) = 0$ and $Q_\gamma(v^{\tilde{\beta}_0}) \leqslant 0$. Thus in any case we have $S_{\omega,\gamma}(v^{\beta_0}) \geqslant d(\omega)$. Now, we have

$$S_{\omega,\gamma}(v) - S_{\omega,\gamma}(v^{\beta_0}) = \frac{1-\beta_0}{2}\|\partial_x v\|_2^2 + (1-\beta_0{}^{-1})\left(\frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}\right),$$

from the expression of Q_γ and $\beta_0 > 1$ it follows that

$$S_{\omega,\gamma}(v) - S_{\omega,\gamma}(v^{\beta_0}) \geqslant \frac{1}{2}(Q_\gamma(v) - Q_\gamma(v^{\beta_0})). \tag{2.30}$$

Therefore, from (2.30), $Q_\gamma(v^{\beta_0}) \leqslant 0$ and $S_{\omega,\gamma}(v^{\beta_0}) \geqslant d(\omega)$ we have

$$Q_\gamma(v) \leqslant -m = 2(S_{\omega,\gamma}(v) - d(\omega)) < 0 \tag{2.31}$$

where m is independent of t since $S_{\omega,\gamma}$ is a conserved quantity.

Conclusion. Finally, thanks to (2.31) and Proposition 2.10, we have

$$\|xu^\alpha(t)\|_2^2 \leqslant -4mt^2 + Ct + \|x\varphi_{\omega,\gamma}^\alpha\|_2^2. \tag{2.32}$$

For t large, the right member of (2.32) becomes negative, thus there exists $T^\alpha < +\infty$ such that

$$\lim_{t\to T^\alpha} \|\partial_x u^\alpha(t)\|_2^2 = +\infty.$$

Since it is clear that $\varphi_{\omega,\gamma}^\alpha \to \varphi_{\omega,\gamma}$ in $H^1(\mathbb{R})$ when $\alpha \to 1$, Theorem 2.2 is proved.

Remark 2.30. It is not hard to see that the set

$$\mathcal{I} = \{v \in H^1(\mathbb{R}); S_{\omega,\gamma}(v) < d(\omega), I_{\omega,\gamma}(v) > 0\}$$

is invariant under the flow of (2.1), and that a solution with initial data belonging to $\mathcal{I}$ is global. Thus using the minimizing character of $\varphi_{\omega,\gamma}$ and performing an analysis in the same way than in [19], it is possible to find a family of initial data in $\mathcal{I}$ approaching $\varphi_{\omega,\gamma}$ in $H^1(\mathbb{R})$ and such that the associated solution of (2.1) exists globally but escaped in finite time from a tubular neighborhood of $\varphi_{\omega,\gamma}$ (see also [10, 17] for an illustration of this approach on a related problem).

2.4 The virial theorem

This section is devoted to the proof of Proposition 2.10.

For $a \in \mathbb{N} \setminus \{0\}$, we define $V^a(x) = \gamma a e^{-\pi a^2 x^2}$. It is clear that $\int_{\mathbb{R}} V^a(x) = \gamma$ and $V^a \rightharpoonup \gamma\delta$ weak-$\star$ in $H^{-1}(\mathbb{R})$ when $a \to +\infty$.

We begin by the construction of approximate solutions : for

$$\begin{cases} i\partial_t u &= -\partial_{xx}u - V^a u - |u|^{p-1}u, \\ u(0) &= u_0, \end{cases} \tag{2.33}$$

and thanks to [6, Proposition 6.4.1], for every $a \in \mathbb{N} \setminus \{0\}$ there exists $T^a > 0$ and a unique maximal solution $u^a \in \mathcal{C}([0,T^a), H^1(\mathbb{R})) \cap \mathcal{C}^1([0,T^a), H^{-1}(\mathbb{R}))$ of (2.33) which satisfies for all $t \in [0,T^a)$

$$E^a(u^a(t)) = E^a(u_0), \tag{2.34}$$
$$\|u^a(t)\|_2 = \|u_0\|_2, \tag{2.35}$$

where $E^a(v) = \dfrac{1}{2}\|\partial_x v\|_2^2 - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} V^a|v|^2dx - \dfrac{1}{p+1}\|v\|_{p+1}^{p+1}$. Moreover, the function $f^a : t \mapsto \int_{\mathbb{R}} x^2|u^a(t,x)|^2dx$ is $\mathcal{C}^2$ by [6, Proposition 6.4.2], and

$$\partial_t f^a = 4\mathrm{Im}\int_{\mathbb{R}} \overline{u^a} x\partial_x u^a dx, \tag{2.36}$$
$$\partial_{tt} f^a = 8Q_\gamma^a(u^a) \tag{2.37}$$

where Q_γ^a is defined for $v \in H^1(\mathbb{R})$ by

$$Q_\gamma^a(v) = \|\partial_x v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}} x(\partial_x V^a)|v|^2dx - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Then, we find estimates on (u^a). Let $M \geqslant \|u_0\|_{H^1(\mathbb{R})}$ (an exact value of M will be precise later). We define

$$t^a = \sup\{t > 0; \|u^a(s)\|_{H^1(\mathbb{R})} \leqslant 2M \text{ for all } s \in [0,t)\}. \tag{2.38}$$

Since u^a satisfies (2.33), we have

$$\sup_{a\in\mathbb{N}\setminus\{0\}} \|\partial_t u^a\|_{L^\infty([0,t^a),H^{-1}(\mathbb{R}))} \leqslant C,$$

and thus for all $t \in [0,t^a)$ and for all $a \in \mathbb{N} \setminus \{0\}$ we get

$$\|u^a(t) - u_0\|_2^2 = 2\mathrm{Re}\int_0^t (u^a(s) - u_0, \partial_t u^a(s))_2 ds \leqslant Ct \tag{2.39}$$

where C depends only on M. Now we have

$$\frac{1}{p+1}(\|u^a\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1}) = \int_0^1 \mathrm{Re}\int_{\mathbb{R}} (u^a - u_0)|su^a + (1-s)u_0|^p dx\, ds$$

which combined with Hölder inequality, Sobolev embeddings, (2.38) and (2.39) gives

$$\frac{1}{p+1}(\|u^a\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1}) \leqslant Ct^{1/2}. \tag{2.40}$$

Moreover, using (2.38), Sobolev embeddings, Gagliardo-Nirenberg inequality and (2.39) we obtain

$$\left| \int_{\mathbb{R}} V^a(|u^a|^2 - |u_0|^2) \right| \leqslant Ct^{1/4}. \tag{2.41}$$

Combining (2.34), (2.35), (2.40) and (2.41) leads to

$$\|u^a(t)\|_{H^1(\mathbb{R})}^2 \leqslant M^2 + C(t^{1/4} + t^{1/2}) \text{ for all } t \in [0, t^a) \text{ and for all } a \in \mathbb{N} \setminus \{0\},$$

and choosing T_M (depending only on M) such that $C(T_M^{1/4} + T_M^{1/2}) = 3M^2$ we obtain for all $a \in \mathbb{N} \setminus \{0\}$ the estimates

$$\begin{aligned} &\|u^a\|_{L^\infty([0,T_M),H^1(\mathbb{R}))} \leqslant 2M, \\ &\|\partial_t u^a\|_{L^\infty([0,T_M),H^{-1}(\mathbb{R}))} \leqslant C. \end{aligned} \tag{2.42}$$

In particular, it follows from (2.42) that $T_M \leqslant t^a$ for all $a \in \mathbb{N} \setminus \{0\}$.

Now we can pass to the limit : thanks to (2.42) there exists $u \in L^\infty([0, T_M), H^1(\mathbb{R}))$ such that for all $t \in [0, T_M)$ we have

$$u^a(t) \rightharpoonup u(t) \text{ weakly in } H^1(\mathbb{R}) \text{ when } a \to +\infty, \tag{2.43}$$

which immediately induces that when $a \to +\infty$,

$$|u^a(t)|^{p-1}u^a(t) \rightharpoonup |u(t)|^{p-1}u(t) \text{ weakly in } H^{-1}(\mathbb{R}). \tag{2.44}$$

In particular, thanks to Sobolev embeddings, we have

$$u^a(t, x) \to u(t, x) \text{ a.e. and uniformly on the compact sets of } \mathbb{R},$$

and it is not hard to see that it permit to show

$$V^a u^a \rightharpoonup u\gamma\delta \text{ weak-}\star \text{ in } H^{-1}(\mathbb{R}). \tag{2.45}$$

Since u^a satisfies (2.33), it follows from (2.43), (2.44) and (2.45) that u satisfies (2.1). Finally, by (2.5) and (2.35), we have

$$u^a \to u \text{ in } \mathcal{C}([0, T_M), L^2(\mathbb{R})),$$

thus, from Gagliardo-Nirenberg inequality and (2.42), we have

$$u^a \to u \text{ in } \mathcal{C}([0, T_M), L^{p+1}(\mathbb{R})),$$

and by (2.4) and (2.34) it follows that

$$u^a \to u \text{ in } \mathcal{C}([0,T_M),H^1(\mathbb{R})). \tag{2.46}$$

We have to prove that the time interval $[0,T_M)$ can be extended as large as we need. Let $0 < T < T_{u_0}$ and

$$M = \sup\{\|u(t)\|_{H^1(\mathbb{R})}, t \in [0,T]\}.$$

If $T_M \geqslant T$, there is nothing left to do, thus we suppose $T_M < T$. >From (2.46) we have $\|u^a(T_M)\|_{H^1(\mathbb{R})} \leqslant M$ for a large enough. By performing a shift of time of length T_M in (2.1) and (2.33) and repeating the first steps of the proof we obtain

$$u^a \to u \text{ in } \mathcal{C}([T_M,2T_M),H^1(\mathbb{R})).$$

Now we reiterate this procedure a finite number of times until we covered the interval $[0,T]$ to obtain

$$u^a \to u \text{ in } \mathcal{C}([0,T],H^1(\mathbb{R})). \tag{2.47}$$

To conclude, we remark that (2.6) follows from the same proof than [6, Lemma 6.4.3] (computing with $\|e^{-\varepsilon|x|^2}xu(t)\|_2^2$ and passing to the limit $\varepsilon \to 0$), thus we have

$$\|xu(t)\|_2^2 = \|xu_0\|_2^2 + 4\int_0^t \operatorname{Im}\int_{\mathbb{R}} \overline{u(s)}x\partial_x u(s)dxds. \tag{2.48}$$

From (2.36), Cauchy-Schwartz inequality and (2.42) we have

$$\partial_t\left(\|xu^a(t)\|_2^2\right) \leqslant C\|xu^a(t)\|_2,$$

which implies that

$$\|xu^a(t)\|_2 \leqslant \|xu_0\|_2 + Ct.$$

Since in addition we have

$$xu^a(t,x) \to xu(t,x) \text{ a.e.},$$

we infer that

$$xu^a(t,x) \rightharpoonup xu(t,x) \text{ weakly in } L^2(\mathbb{R}).$$

Recalling that

$$\partial_x u^a \to \partial_x u \text{ strongly in } L^2(\mathbb{R})$$

we can pass to the limit in (2.48) to have

$$\|xu^a(t)\|_2 \to \|xu(t)\|_2.$$

On the other side, since we have (2.37) and (2.47), we get (2.7).

Remark 2.31. Our method of approximation is inspired of the one developed in [8] by Cazenave and Weissler to prove the local well-posedness of the Cauchy problem for nonlinear Schrödinger equations. Actually, slight modifications in our proof of Proposition 2.10 would permit to give an alternative proof of Proposition 2.3.

2.5 Numerical results

In this Section, we use numerical simulations to complement the rigorous theory on stability and instability of the standing waves of (2.1). Our approach here is similar to the one in [9]. In order to study stability under radial perturbations, we use the initial condition

$$u_0(x) = (1 + \delta_p)\varphi_{\omega,\gamma}(x). \tag{2.49}$$

In order to study stability under non-radial (asymmetric) perturbations, we use the initial condition

$$u_0(x) = \varphi_{\omega,\gamma}(x - \delta_c), \tag{2.50}$$

when δ_c is the lateral shift of the initial condition. In some cases (when the standing wave has a negative slope and the linearized problem has two negative eigenvalues), we use the initial condition

$$u_0(x) = (1 + \delta_p)\varphi_{\omega,\gamma}(x - \delta_c). \tag{2.51}$$

2.5.1 Stability in $H^1_{\mathrm{rad}}(\mathbb{R})$

Strength of radial stability

When $\gamma > 0$, the standing waves are known to be stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ for $1 < p \le 5$. The rigorous theory, however, does not address the issue of the *strength of radial stability*. This issue is of most interest in the case $p = 5$, which is unstable when $\gamma = 0$.

For $\delta_p > 0$, it is useful to define

$$F(\delta_p) = \max_{t\ge 0}\left\{\frac{\max_x |u(x,t)| - \max_x \varphi_{\omega,\gamma}}{\max_x \varphi_{\omega,\gamma}}\right\} \tag{2.52}$$

as a measure of the strength of radial stability. Figure 2.2 shows the normalized values $\max_x |u|/\max_x \varphi_{\omega,\gamma}$ as a function of t, for the initial condition (2.49) with $\omega = 4$ and $\gamma = 1$. When $p = 3$, a perturbation of $\delta_p = 0.01$ induces small oscillations and $F(0.01) = 1.9\%$. Therefore, roughly speaking, a 1% perturbation of the initial condition leads to a maximal deviation of 2%. A larger perturbation of $\delta_p = 0.08$ causes the magnitude of the oscillations to increase approximately by the same ratio, so that $F(0.08) = 15\%$. Using the same perturbations with $p = 5$, however, leads to significantly larger deviations. Thus, $F(0.01) = 8.8\%$, i.e., more than 4 times bigger than for $p = 3$, and $F(0.08) = 122\%$, i.e., more than 8 times than for $p = 3$.

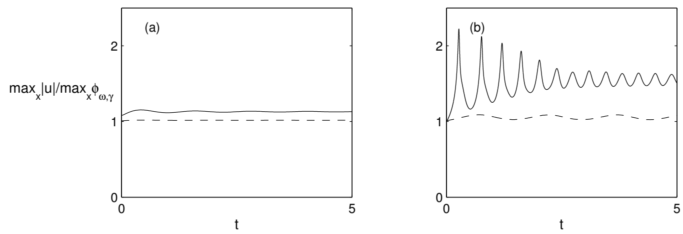

FIGURE 2.2 - $max_x|u|/max_x\varphi_{\omega,\gamma}$ as a function of t for $\omega = 4, \gamma = 1$, $\delta_p = 0.01$ (dashed line) and $\delta_p = 0.08$ (solid line). (a) $p = 3$ (b) $p = 5$.

In [9, 35], Fibich, Sivan and Weinstein observed that the strength of radial stability is related to the magnitude of slope $\partial_\omega ||\varphi_{\omega,\gamma}||_2^2$, so that the larger $\partial_\omega ||\varphi_{\omega,\gamma}||_2^2$, the "more stable" the solution is. Indeed, numerically we found that when $\omega = 4$, $\partial_\omega ||\varphi_{\omega,\gamma}||_2^2$ is equal to 1.0 for $p = 3$ and 0.056 for $p = 5$.

Since when $\gamma = 0$, the slope is positive for $p < 5$ but zero for $p = 5$, for $\gamma > 0$ the slope is smaller in the critical case than in the subcritical case. Therefore, we make the following informal observation:

Observation 2.1. *Radial stability of the standing waves of* (2.1) *with* $\gamma > 0$ *is "weaker" in the critical case* $p = 5$ *than in the subcritical case* $p < 5$.

Clearly, this difference would be more dramatic at smaller (positive) values of γ. Indeed, if in the simulation of Figure 2.2 with $\delta_p = 0.01$ we reduce γ from 1 to 0.5 and then to 0.1, this has almost no effect when $p = 3$, where the value of F slightly increases from 1.9% to 2.1% and to 2.5%, respectively, see Figure 2.3a. However, if we repeat the same simulations with $p = 5$, then reducing the value of γ has a much larger effect, see Figure 2.3b, where F increases from 8.9% for $\gamma = 1$ to 24% for $\gamma = 0.5$. Moreover, when we further reduced γ to 0.1, the solution seems to undergo collapse.[1] This implies that when $p = 5$ and $\gamma > 0$, the standing wave is stable, yet it can collapse under a sufficiently large perturbation.

[1] Clearly, one cannot use numerics to determine that a solution becomes singular, as it is always possible that collapse would be arrested at some higher focusing levels.

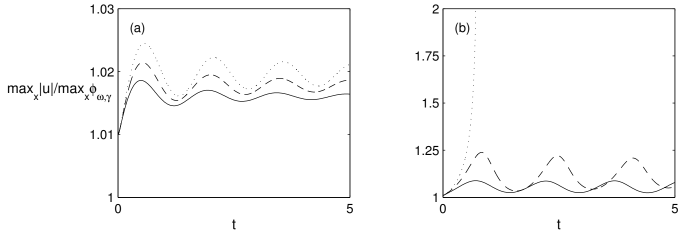

Figure 2.3 - $max_x|u|/max_x\varphi_{\omega,\gamma}$ as a function of t for $\omega = 4$, $\delta_p = 0.01$, and $\gamma = 1$ (solid line), $\gamma = 0.5$ (dashed line) and $\gamma = 0.1$ (dots). (a) $p = 3$ (b) $p = 5$.

Characterization of radial instability for $3 < p < 5$ and $\gamma < 0$

We consider the subcritical repulsive case $p = 4$ and $\gamma = -1$. In this case, there is threshold ω_2 such that $\varphi_{\omega,\gamma}$ is stable for $\omega > \omega_2$ and unstable for $\omega < \omega_2$. By numerical calculation we found that $\omega_2(p = 4, \gamma = -1) \approx 0.82$. Accordingly, we chose two representative values of ω: $\omega = 0.5$ in the unstable regime, and $\omega = 2$ in the stable regime.

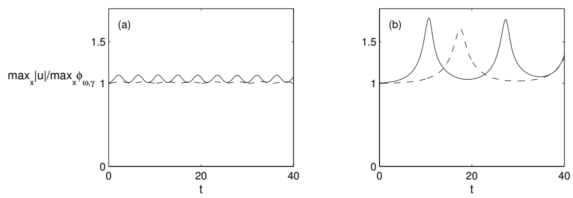

Figure 2.4 - $max_x|u|/max_x\varphi_{\omega,\gamma}$ as a function of t for $p = 4, \gamma = -1$, $\delta_p = 0.001$ (dashed line) and $\delta_p = 0.005$ (solid line). (a) $\omega = 2$; (b) $\omega = 0.5$.

Figure 2.4a demonstrates the stability for $\omega = 2$. Indeed, reducing the perturbation from $\delta_p = 0.005$ to 0.001 results in reduction of the relative magnitude of the oscillations by roughly five times, from $F(0.005) \approx 10\%$ to $F(0.001) \approx 2\%$. The dynamics in the unstable case $\omega = 0.5$ is also oscillatory, see Figure 2.4b. However, in this case $F(0.005) = 79\%$, i.e., eight times larger than for $\omega = 2$. More importantly, unlike the stable case, a perturbation of $\delta_p = 0.001$ does not result in a reduction of the relative magnitude of the oscillations by ≈ 5. In fact, the relative magnitude of the oscillations descreases only to $F(0.001) = 66\%$.

In the homogeneous NLS, unstable standing waves perturbed with $\delta_p > 0$ always undergo collapse. Since, however, for $p = 4$ it is impossible to have collapse, an interesting question is the nature of the instability in the unstable region $\omega < \omega_2$. In Figure 2.4b we already saw that $max|u(x,t)|$ undergoes oscillations. In order to better understand the nature of this unstable oscillatory dynamics, we plot in Figure 2.5 the spatial profile of $|u(x,t)|$ at various values of t. In addition, at each t

we plot $\phi_{\omega^*(t),\gamma}(x)$, where $\omega^*(t)$ is determined from the relation

$$\max_x \phi_{\omega^*(t),\gamma}(x) = \max_x |u(x,t)|.$$

Since the two curves are nearly indistinguishable (especially in the central region), this shows that the unstable dynamics corresponds to "movement along the curve $\phi_{\omega*(t)}$".

In Figure 2.6 we see that $\omega^*(t)$ undergoes oscillations, in accordance with the oscillations of $\max_x |u|$. Furthermore, as one may expect, collapse is arrested only when $\omega^*(t)$ reaches a value (≈ 2.86) which is in the stability region (i.e., above ω_2).

Observation 2.2. *When $\gamma < 0$ and $3 < p < 5$, the instability in $H^1_{\text{rad}}(\mathbb{R})$ is a "finite width instability", i.e., the solution narrows down along the curve $\phi_{\omega*(t),\gamma}$ until it "reaches" a finite width in the stable region $\omega > \omega_2$, at which point collapse is arrested.*

Note that this behavior was already observed in [9], Fig 19. Therefore, more generally, we conjecture that

Observation 2.3. *When the slope is negative (i.e., $\partial_\omega ||\varphi_{\omega,\gamma}||_2^2 < 0$), then the symmetric perturbation (2.49) with $0 < \delta_p \ll 1$ leads to a finite-width instability in the subcritical case, and to a finite-time collapse in the critical and supercritical cases.*

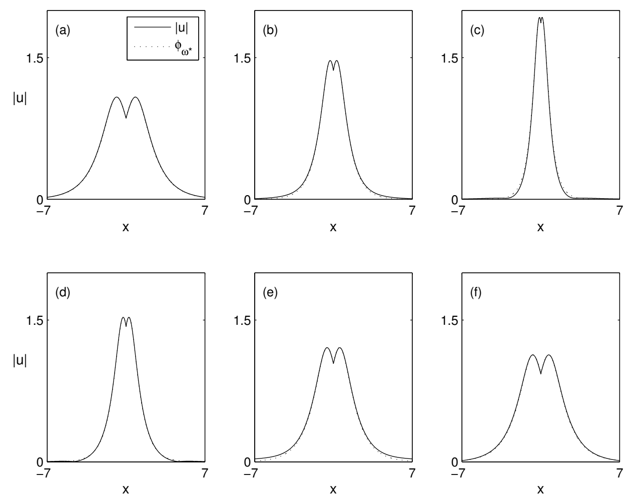

FIGURE 2.5 - $|u(x,t)|$ (solid line) and $\phi_{\omega*(t)}(x)$ (dots) as a function of x for the simulation of Fig. 2.4b with $\delta_p = 0.005$. (a) $t = 0$ ($\omega^* = 0.508$) (b) $t = 9$ ($\omega^* = 1.27$) (c) $t = 10.69$ ($\omega^* = 2.86$) (d) $t = 12$ ($\omega^* = 1.43$) (e) $t = 15$ ($\omega^* = 0.706$) (f) $t = 20$ ($\omega^* = 0.58$).

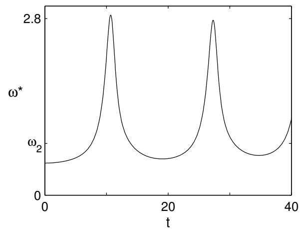

FIGURE 2.6 - ω^* as a function of t for the simulation of Fig 2.5.

Supercritical case $(p > 5)$

We recall that when $\gamma > 0$ and $p > 5$, the standing wave is stable for $\gamma^2/4 < \omega < \omega_1$ and unstable for $\omega_1 < \omega$. When $\gamma < 0$ and $p > 5$ the standing wave is strongly unstable under radial perturbations for any ω, i.e., an infinitesimal perturbation can lead to collapse.
Figure 2.7 shows the behavior of perturbed solutions for $p = 6$ and $\omega = 1$. As predicted by the theory, when $\delta_p = 0.001$, the solution blows up for $\gamma = -1$ and $\gamma = 0$, but undergoes small oscillations (i.e., is stable) for $\gamma = 1$. Indeed, we found numerically that $\omega_1(p = 6, \gamma = 1) \approx 2.9$, so that the standing wave is stable for $\omega = 1$. However, when we increase the perturbation to $\delta_p = 0.1$, the solution with $\gamma = 1$ also seems to undergo collapse. This implies that when $p > 5, \gamma > 0$ and $\omega < \omega_1$ the standing wave is stable, yet it can collapse under a sufficiently large perturbation. In order to find the type of instability for $\gamma > 0$ and $\omega > \omega_1$, we solve the NLS (2.1) with $p = 6$, $\gamma = 1$ and $\omega = 4$. In this case, $\delta_p = 0.001$ seems to lead to collapse, see Figure 2.8, suggesting a strong instability for $p > 5$, $\gamma > 0$ and $\omega > \omega_1$. Therefore, we make the following informal observation:

Observation 2.4. *If a standing wave of* (2.1) *with* $p > 5$ *is unstable in* $H^1_{\mathrm{rad}}(\mathbb{R})$*, then the instability is strong.*

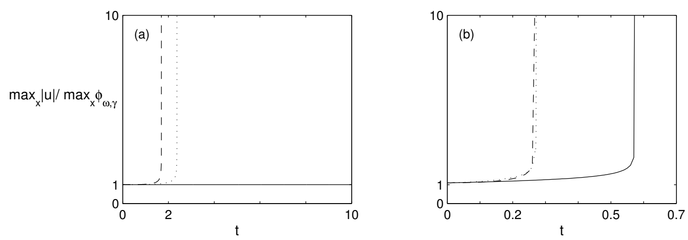

FIGURE 2.7 - $\max_x |u(x,t)| / \max_x \varphi_{\omega,\gamma}$ as a function of t for $p = 6, \omega = 1$ and $\gamma = -1$ (dashed line), $\gamma = 0$ (dots), $\gamma = +1$ (solid line). (a) $\delta_p = 0.001$ (b) $\delta_p = 0.1$.

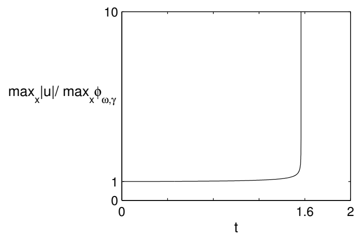

FIGURE 2.8 - $\max_x |u(x,t)|/\max_x \varphi_{\omega,\gamma}$ as a function of t for $p = 6, \omega = 4$, $\gamma = 1$ and $\delta_p = 0.001$.

2.5.2 Stability under non-radial perturbations

Stability for $1 < p < 5$ and $\gamma > 0$

Figure 2.9 shows the evolution of the solution when $p = 3, \gamma = 1, \omega = 1$ and $\delta_c = 0.1$. The peak of the solution moves back towards $x = 0$ very quickly (around $t \approx 0.003$) and stays there at later times. Subsequently, the solution converges to the bound state $\phi_{\omega^*=0.995}$. This convergence starts near $x = 0$ and spreads sideways, accompanied by radiation of the excess power $||u_0||_2^2 - ||\phi_{\omega^*=0.995}||_2^2 \cong 2.00 - 1.99 = 0.01$. In Fig 2.10 we repeat this simulation with a larger shift of $\delta_c = 0.5$. The overall dynamics is similar: The solution peak moves back to $x = 0$, and the solution converges (from the center outwards) to $\phi_{\omega^*=0.905}$. In this case, it takes longer for the maximum to return to $x = 0$ (at $t \approx 0.11$), and more power is radiated in the process ($||u_0||_2^2 - ||\phi_{\omega^*=0.905}||_2^2 \cong 2.00 - 1.81 = 0.19$. We verified that the "non-smooth" profiles (e.g., at $t = 0.2$) are not numerical artifacts.

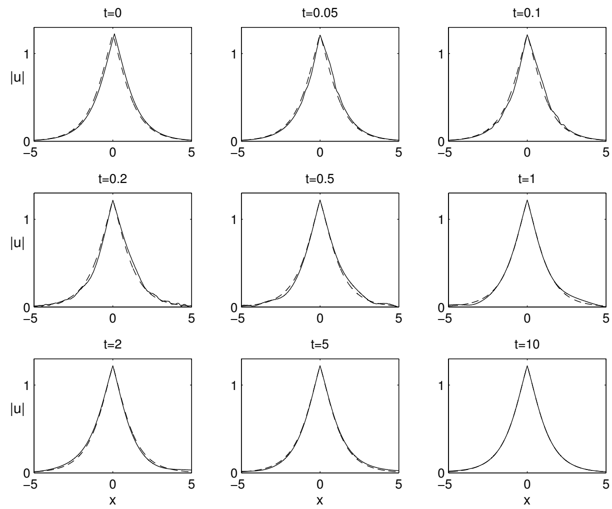

FIGURE 2.9 - $|u(x,t)|$ (solid line) and $\phi_{\omega*=0.995}(x)$ (dashed line) as a function of x. Here, $p = 3, \omega = 1, \gamma = 1$ and $\delta_c = 0.1$.

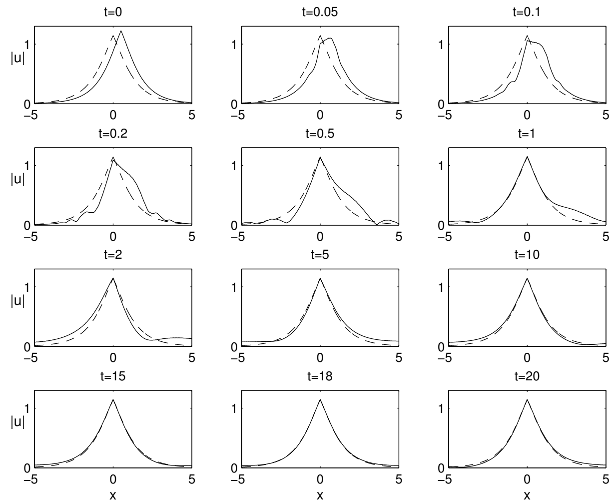

FIGURE 2.10 - Same as Fig 2.9 with $\delta_c = 0.5$ and $\omega^* = 0.905$.

Drift instability for $1 < p \leq 3$ and $\gamma < 0$

Figure 2.11 shows the evolution of the solution for $p = 3, \gamma = -1, \omega = 1$ and $\delta_c = 0.1$. Unlike the attractive case with the same parameters (Figure 2.9), as a result of this small initial shift to the right, nearly all the power flows from the left side of the defect ($x < 0$) to the right side ($x > 0$), see Figure 2.12a, so that by $t \approx 3$, $\approx 90\%$ of the power is in the right side. Subsequently, the right component moves to the right at a constant speed (see Fig 2.12b) while assuming the *sech* profile of the homogeneous NLS bound state (see Fig 2.11 at t=8); the left component also drifts away from the defect.

We thus see that

Observation 2.5. *When $1 < p \leq 3$, the standing waves are stable under shifts in the attractive case, but undergo a drift instability away from the defect in the repulsive case.*

We note that a similar behavior was observed in the subcritical NLS with a periodic nonlinearity, see [9], Section 5.1.

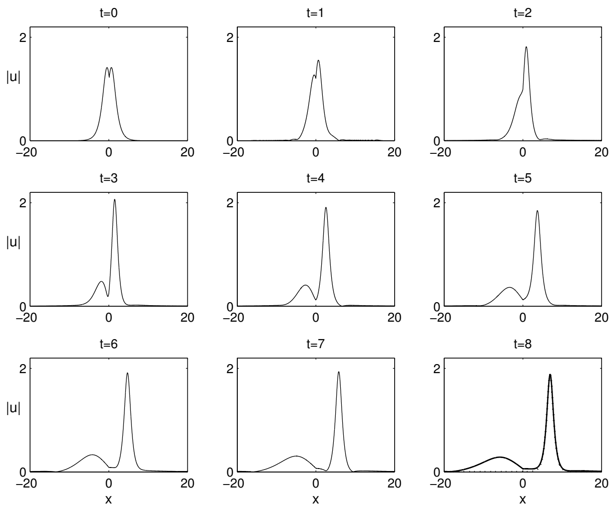

FIGURE 2.11 - $|u(x,t)|$ (solid line) as a function of x. Here $p = 3, \gamma = -1, \omega = 1$ and $\delta_c = 0.1$. Dotted line at $t = 8$ is $\sqrt{2\omega^*} sech(\sqrt{\omega^*}(x - x^*))$ with $\omega^* = 1.768$ and $x^* \approx 7$.

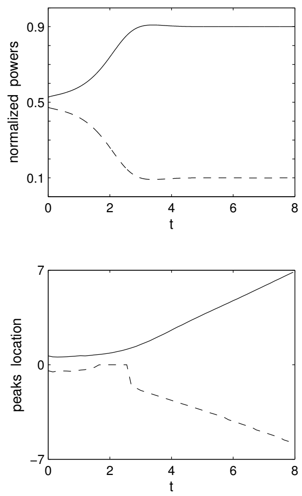

FIGURE 2.12 - (a) The normalized powers $\int_0^\infty |u|^2 dx / \int_{-\infty}^\infty |u_0|^2 dx$ (solid line) and $\int_{-\infty}^0 |u|^2 dx / \int_{-\infty}^\infty |u_0|^2 dx$ (dashed line), and (b) location of $\max_{0 \le x} |u(x,t)|$ (solid line) and of $\max_{x \le 0} |u(x,t)|$ (dashed line), for the simulation of Figure 2.11.

Drift and finite-width instability for $3 < p < 5$ and $\gamma < 0$

In Figure 2.4b, Figure 2.5, and Figure 2.6 we saw that when $p = 4$, $\gamma = -1$, $\omega = 0.5$, and $\delta_p = 0.005$, the solution undergoes a finite-width instability in $H^1_{\text{rad}}(\mathbb{R})$. In Figures 2.13 and 2.14 we show the dynamics (in $H^1(\mathbb{R})$) when we add a small shift of $\delta_c = 0.1$. In this case, the (larger) right component undergoes a combination of a drift instability and a finite-width instability, whereas the (smaller) left component undergoes a drift instability. Therefore, we make the following observation

Observation 2.6. *When $3 < p < 5$, $\gamma^2/4 < \omega < \omega_2$ and $\gamma < 0$, the standing waves undergo a combined drift and finite-width instability.*

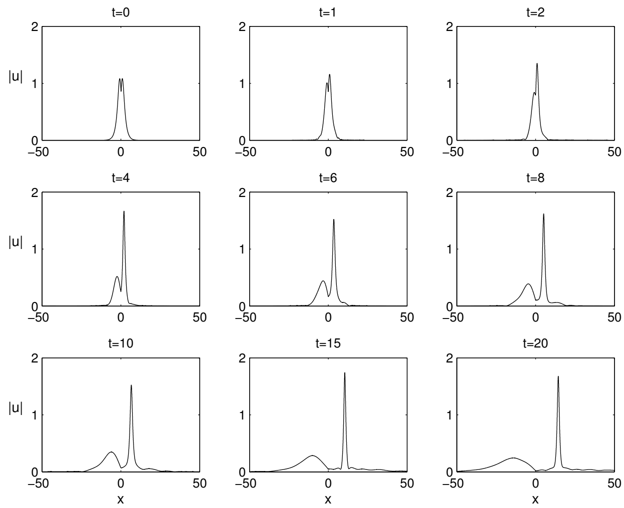

Figure 2.13 - $u(x, t)$ as a function of x. Here $p = 4$, $\gamma = -1$, $\omega = 0.5$, $\delta_p = 0.005$, and $\delta_c = 0.1$

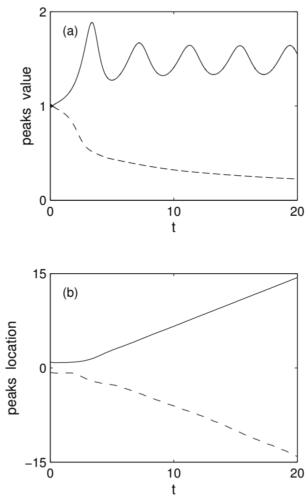

FIGURE 2.14 - (a) The value, and (b) the location, of the right peak $\max_{0\leq x}|u(x,t)|$ (solid line) and left peak $\max_{x\leq 0}|u(x,t)|$ (dashed line), for the simulation of Figure 2.13.

Drift and strong instability for $5 \leq p$ and $\gamma < 0$

In Figures 2.15 and 2.16 we show the solution of the NLS (2.1) with $p = 6$, $\gamma = -1$ and $\omega = 1$, for the initial condition (2.51) with $\delta_c = 0.2$ and $\delta_p = 0.001$. As predicted by the theory, this strongly unstable solution undergoes collapse. Note, however, that, in parallel, the solution also undergoes a drift instability. We thus see that

Observation 2.7. *In the critical and supercritical repulsive case, the standing waves collapse while undergoing a drift instability away from the defect.*

Note that a similar behavior was observed in [9], Section 5.2.

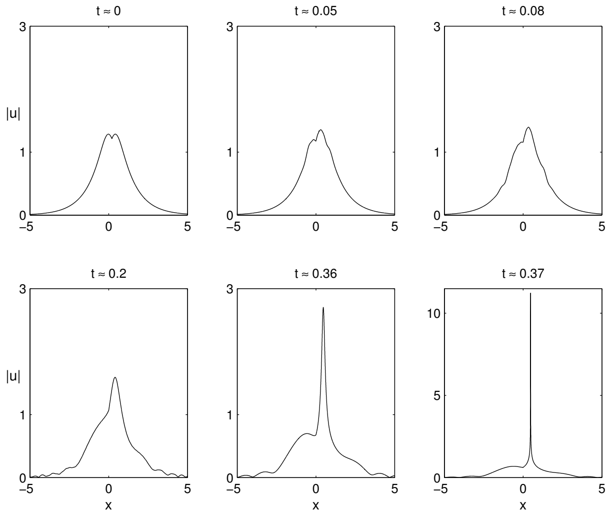

Figure 2.15 - $|u(x,t)|$ as a function of x, at various values of t. Here, $p = 6$, $\gamma = -1$, $\omega = 1$, $\delta_c = 0.2$ and $\delta_p = 0.001$.

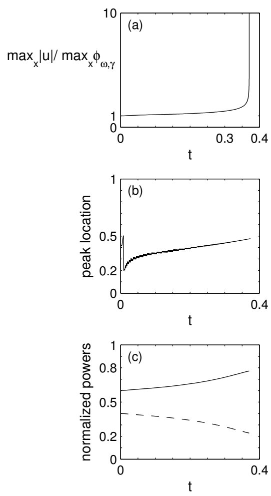

Figure 2.16 - (a) $\max_x |u(x,t)| / \max_x \varphi_{\omega,\gamma}$ (b) location of $\max_x |u(x,t)|$ and (c) The normalized powers $\int_0^\infty |u|^2 dx / \int_{-\infty}^{\infty} |u_0|^2 dx$ (solid line) and $\int_{-\infty}^0 |u|^2 dx / \int_{-\infty}^{\infty} |u_0|^2 dx$ (dashed line), for the solution of Fig. 2.15.

2.5.3 Numerical Methods

We solve the NLS (2.1) using fourth-order finite differences in x and second-order implicit Crack-Nicholson scheme in time. Clearly, the main question is how to discretize the delta potential at $x = 0$. Recall that in continuous case

$$\lim_{x\to 0^+} \partial_x u(x) - \lim_{x\to 0^-} \partial_x u(x) = -\gamma u(0).$$

Discretizing this relation with $\mathcal{O}(h^2)$ accuracy gives

$$\frac{u(2h) - 4u(h) + 3u(0)}{2h} - \frac{-u(-2h) + 4u(-h) - 3u(0)}{2h} = -\gamma u(0),$$

when h is the spatial grid size. By rearrangement of the terms we get the equation

$$-u(2h) + 4u(h) + [2h\gamma - 6]u(0) + 4u(-h) - u(-2h) = 0. \tag{2.53}$$

When we simulate symmetric perturbations (section 2.5.1), we enforce symmetry by solving only on half space $[0, +\infty)$. In this case, because of the symmetry condition $u(-x) = u(x)$, (2.53) becomes

$$[2h\gamma - 6]u(0) + 8u(h) - 2u(2h) = 0.$$

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Chapitre 3

A note on Berestycki-Cazenave's classical instability result for nonlinear Schrödinger equations

Abstract. In this note we give an alternative, shorter proof of the classical result of Berestycki and Cazenave on the instability by blow-up for the standing waves of some nonlinear Schrödinger equations.

3.1 Introduction

In 1981, in a celebrated note [1], Berestycki and Cazenave studied the instability of standing waves for the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0 \tag{3.1}$$

where $u = u(t,x) \in \mathbb{C}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $p > 1$. A standing wave is a solution of (3.1) of the form $e^{i\omega t}\varphi(x)$ with $\varphi \in H^1(\mathbb{R}^N)$ and $\omega > 0$. Thus φ is solution of

$$-\Delta\varphi + \omega\varphi = |\varphi|^{p-1}\varphi, \quad \varphi \in H^1(\mathbb{R}^N). \tag{3.2}$$

We say that $\varphi \in H^1(\mathbb{R}^N)$ is a ground state solution of (3.2) if it satisfies

$$\tilde{S}(\varphi) = \inf\{\tilde{S}(v); v \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of (3.2) }\}$$

where $\tilde{S}$ is defined for $v \in H^1(\mathbb{R}^N)$ by

$$\tilde{S}(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1}dx.$$

In [1] it is shown that if $1 + \frac{4}{N} < p < 1 + \frac{4}{N-2}$ when $N \geqslant 3$ and $1 + \frac{4}{N} < p < +\infty$ when $N = 1, 2$, then any standing wave associated with a ground state solution φ of (3.2) is unstable by blow up. More precisely, there exists $(\varphi_n) \subset H^1(\mathbb{R}^N)$ such that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and the corresponding maximal solution u_n of (3.1) with $u_n(0) = \varphi_n$ blows up in finite time.

The result and perhaps more the methods introduced in [1] still have a deep influence on the field of instability for nonlinear Schrödinger and related equations. In particular the idea of defining appropriate invariant sets and how to use them to establish the blow-up. We should mention that in [1] more general nonlinearities were considered. The paper [1] is only a short note which contains the main ideas but no proofs. For the special nonlinearity $|u|^{p-1}u$ these proofs can be found in [5]. For the general case it seems that the extended version [2] of [1] has remained unpublished so far.

The aim of the present note is to present an alternative, shorter proof of the result of [1] for general nonlinearities. Also the instability of the standing waves is proved under slightly weaker assumptions. Before stating our result we need to introduce some notations. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be an odd function extended to $\mathbb{C}$ by setting $g(z) = g(|z|)z/|z|$ for $z \in \mathbb{C} \setminus \{0\}$. Equation (3.1) now becomes

$$iu_t + \Delta u + g(u) = 0 \tag{3.3}$$

and correspondingly for the ground states we have

$$-\Delta\varphi + \omega\varphi = g(\varphi). \tag{3.4}$$

For $z \in \mathbb{C}$ let $G(z) := \int_0^{|z|} g(s)ds$. We assume

(A_0) The function g satisfies

- (a) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.
- (b) $\lim_{s\to 0} \frac{g(s)}{s} = 0$.
- (c) when $N \geqslant 3$, $\lim_{s\to +\infty} g(s)s^{-\frac{N+2}{N-2}} = 0$;
  when $N = 2$, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|g(s)| \leqslant C_\alpha e^{\alpha s^2}$ for all $s > 0$.

(A_1) The function $h(s) := (sg(s) - 2G(s))s^{-(2+4/N)}$ is strictly increasing on $(0, +\infty)$ and $\lim_{s\to 0} h(s) = 0$.

(A_2) There exist $C > 0$ and $\alpha \in [0, \frac{4}{N-2})$ if $N \geqslant 3$, $\alpha \in [0, \infty)$ if $N = 2$, such that

$$|g(s) - g(t)| \leqslant C(1 + |s|^\alpha + |t|^\alpha)|t - s|$$

for all $s, t \in \mathbb{R}$. If $N = 1$ we assume that for every $M > 0$, there exists $L(M) > 0$ such that

$$|g(s) - g(t)| \leqslant L(M)|s - t|$$

for all $s, t \in \mathbb{R}$ such that $|s| + |t| \leqslant M$.

Finally we define for $v \in H^1(\mathbb{R}^N)$ the functional

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \int_{\mathbb{R}^N} G(v)dx$$

and set

$$m := \inf\{S(v); v \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of (3.4) }\}.$$

Our main result is

Theorem 3.1. *Assume that $(A_0) - (A_2)$ hold and let φ be a ground state solution of (3.4), i.e. a solution of (3.4) such that $S(\varphi) = m$. Then for every $\varepsilon > 0$ there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u_0 - \varphi\|_{H^1(\mathbb{R}^N)} < \varepsilon$ and the solution u of (3.3) with $u(0) = u_0$ satisfies*

$$\lim_{t \to T_{u_0}} \|\nabla u(t)\|_2 = +\infty \textit{ with } T_{u_0} < +\infty.$$

From [3, 4] we know that assumption (A_0) is almost necessary to guarantee the existence of a solution for (3.4). Assumption (A_1) is a weaker version of the assumption (H.1) in [1]. An assumption of this type, on the growth of g, is necessary since it is known from [6] that when $g(u) = |u|^{p-1}u$ with $1 < p < 1 + \frac{4}{N}$ the standing waves associated with the ground states are, on the contrary, orbitally stable. Assumption (A_2) is purely technical and is aimed at ensuring the local well-posedness of the Cauchy problem for (3.3). It replaces assumption (H.2) in [1]. Indeed, in [1] the authors were using the results of Ginibre and Velo [8] for that purpose. Since [1] has been published, advances have been done in the study of the Cauchy problem (see [5, 7] and the references therein). In particular, under our condition (A_2), for all $u_0 \in H^1(\mathbb{R}^N)$ there exist $T_{u_0} > 0$ and a unique solution $u \in \mathcal{C}([0, T_{u_0}), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1([0, T_{u_0}), H^{-1}(\mathbb{R}^N))$ of (3.3) such that $\lim_{t \to T_{u_0}} \|\nabla u(t)\|_2 = +\infty$ if $T_{u_0} < +\infty$. Furthermore, the following conservation properties hold : for all $t \in [0, T_{u_0})$ we have

$$S(u(t)) = S(u_0), \tag{3.5}$$
$$\|u(t)\|_2 = \|u_0\|_2. \tag{3.6}$$

Finally, if $xu_0 \in L^2(\mathbb{R}^N)$, the function $f : t \mapsto \|xu(t)\|_2^2$ is $\mathcal{C}^2$ and we have the virial identity

$$\partial_{tt} f(t) = 8Q(u(t)), \tag{3.7}$$

where Q is defined for $v \in H^1(\mathbb{R}^N)$ by

$$Q(v) := \|\nabla v\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N} (g(|v|)|v| - 2G(v))dx.$$

The proofs of instability in [1] and here share some elements, in particular the introduction of sets invariant under the flow. The main difference lies in the variational characterization of the ground states which is used to define the invariant sets and how to derive this characterization.

In [1] it is shown that a ground state of (3.4) can be characterized as a minimizer of S on the constraint

$$M := \{v \in H^1(\mathbb{R}^N) \setminus \{0\}, Q(v) = 0\}.$$

To show this characterization, S is directly minimized on M. Additional assumptions (see (H.1) in [1]) are necessary at this step to insure that the minimizing sequences are bounded. Once the existence of a minimizer for S on M has been established, one has to get rid of the Lagrange multiplier, namely to prove that it is zero. There, a stronger version of (A_0), requiring in particular $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and a control on $g'(s)$ at infinity, is necessary along with tedious calculations.

Having established that the ground states of (3.4) minimize S on M, Berestycki and Cazenave show that the set

$$K := \{v \in H^1(\mathbb{R}^N), S(v) < m \text{ and } Q(v) < 0\}$$

is invariant under the flow of (3.3) and that one can choose in K an initial data, arbitrarily close to the ground state, for which the blow-up occurs.

In our approach we characterize the ground states as minimizers of S on

$$\mathscr{M} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\}; Q(v) = 0, I(v) \leqslant 0\},$$

where $I(v)$ is defined for $v \in H^1(\mathbb{R}^N)$ by

$$I(v) := \|\nabla v\|_2^2 + \omega\|v\|_2^2 - \int_{\mathbb{R}^N} g(|v|)|v|dx$$

and correspondingly our invariant set is

$$\{v \in H^1(\mathbb{R}^N), S(v) < m, Q(v) < 0 \text{ and } I(v) < 0\}.$$

The dominant feature of our approach, which also explains why our assumptions on g are weaker than in [1] is that we never explicitly solve a minimization problem.

At the heart of our approach is an additional characterization of the ground states as being at a mountain pass level for S. This characterization was derived in [10] for $N \geqslant 2$ and in [11] for $N = 1$. We also strongly benefit from recent techniques developed by several authors [12, 13, 14, 15, 16, 17] where minimization approches using two constraints have been introduced.

3.2 Proof of Theorem 3.1

We first prove the existence of ground states and the fact that they correspond to minimizers of S on the Nehari manifold.

Lemma 3.1. *Assume that* (A_0) *and* (A_1) *hold. Then (3.4) admits a ground state solution. Furthermore, the ground states solutions of (3.4) are minimizers for*

$$d(\omega) := \inf \left\{ S(v); v \in H^1(\mathbb{R}^N) \setminus \{0\}, I(v) = 0 \right\}.$$

Before proving Lemma 3.1, we prove a technical result.

Lemma 3.2. *Assume that* (A_0) *and* (A_1) *hold. Then the nonlinearity* g *satisfies*

$$\frac{g(s)}{s} \text{ is increasing for } s > 0. \tag{3.8}$$

$$\frac{g(s)}{s} \to +\infty \text{ as } s \to +\infty. \tag{3.9}$$

Proof of Lemma 3.2. From the definition of $h(s)$ we have

$$\frac{g(s)}{s} = s^{4/N} h(s) + \frac{2G(s)}{s^2}. \tag{3.10}$$

Furthermore, for $s > 0$

$$\frac{\partial}{\partial s}\left(\frac{G(s)}{s^2}\right) = \frac{s(sg(s) - 2G(s))}{s^4} > 0 \tag{3.11}$$

where the last inequality follows from (A_1). Thus, combining (3.10), (3.11) and (A_1) we get (3.8) and (3.9). $\square$

Proof of Lemma 3.1. It follows from Lemma 3.2 that

(P) There exists $s_0 > 0$ such that

– if $N \geqslant 2$, then $\frac{1}{2}\omega s_0^2 < G(s_0)$;
– if $N = 1$, then $\frac{1}{2}\omega s^2 > G(s)$ for $s \in (0, s_0)$, $\frac{1}{2}\omega s_0^2 = G(s_0)$ and $\omega s_0 < g(s_0)$.

Now, from [3, Théorème 1] and [4, Theorem 1] we know that the conditions (A_0) and (P) are sufficient to insure the existence of a ground state.

If v is a solution of (3.4), then $S'(v)v = I(v) = 0$; therefore, to prove the lemma it is enough to show that $d(\omega) \geqslant m$. From [10, 11] we know that under (A_0) and (P) the functional S has a mountain pass geometry. More precisely, if we set

$$\Gamma := \{\chi \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)); \chi(0) = 0, S(\chi(1)) < 0\},$$

then $\Gamma \neq \emptyset$ and

$$c := \inf_{\chi\in\Gamma} \max_{t\in[0,1]} S(\chi(t)) > 0.$$

In addition it is shown[1] in [10, 11] that

$$m = c.$$

Namely the *mountain pass level* c corresponds to the ground state level m. Now it is well-known that (3.8) ensure that if $v \in H^1(\mathbb{R}^N)$ satisfies $I(v) = 0$ then $t \mapsto S(tv)$ achieves its unique maximum on $[0, +\infty)$ at $t = 1$. Also (3.9) shows that $\lim_{t\to+\infty} S(tv) = -\infty$. From the definition of c, it implies that $c \leqslant S(v)$ for all $v \in H^1(\mathbb{R}^N)$ such that $I(v) = 0$. Hence we have

$$d(\omega) \geqslant c,$$

and combined with the fact that $m = c$ it ends the proof. □

Now we investigate the behavior of the functionals under some rescaling

Lemma 3.3. *Assume that (A_0) and (A_1) hold. For $\lambda > 0$ and $v \in H^1(\mathbb{R}^N)$, we define $v^\lambda(\,\cdot\,) := \lambda^{\frac{N}{2}} v(\lambda\,\cdot\,)$. We suppose $Q(v) \leqslant 0$. Then there exists $\lambda_0 \leqslant 1$ such that*

(i) $Q(v^{\lambda_0}) = 0$,

(ii) $\lambda_0 = 1$ *if and only if* $Q(v) = 0$,

(iii) $\frac{\partial}{\partial\lambda} S(v^\lambda) > 0$ *for* $\lambda \in (0, \lambda_0)$ *and* $\frac{\partial}{\partial\lambda} S(v^\lambda) < 0$ *for* $\lambda \in (\lambda_0, +\infty)$,

(iv) $\lambda \mapsto S(v^\lambda)$ *is concave on* $(\lambda_0, +\infty)$,

(v) $\frac{\partial}{\partial\lambda} S(v^\lambda) = \frac{1}{\lambda} Q(v^\lambda)$.

[1] In fact, the results of [10, 11] are proved only for real valued functions; however, it is not hard to see that they can be extended to the complex case (see [9, Lemma 14]).

Proof of Lemma 3.3. Easy computations lead to

$$\begin{aligned}\frac{\partial}{\partial\lambda}S(v^\lambda) &= \frac{1}{\lambda}Q(v^\lambda)\\ &= \lambda\left(\|\nabla v\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N}\lambda^{-(N+2)}\left(\lambda^{\frac{N}{2}}g(\lambda^{\frac{N}{2}}|v|)|v| - 2G(\lambda^{\frac{N}{2}}v)\right)dx\right),\end{aligned}$$

and recalling from (A_1) that the function $h(s) := (sg(s)-2G(s))s^{-(2+4/N)}$ is strictly increasing on $[0,+\infty)$, (i), (ii), (iii) and (v) follow easily. To see (iv), we remark that since

$$\left(\|\nabla v\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N}\lambda^{-(N+2)}\left(\lambda^{\frac{N}{2}}g(\lambda^{\frac{N}{2}}|v|)|v| - 2G(\lambda^{\frac{N}{2}}v)\right)dx\right) < 0$$

on $(\lambda_0,+\infty)$, we infer from (A_1) that $\frac{\partial}{\partial\lambda}S(v^\lambda)$ is strictly decreasing on $(\lambda_0,+\infty)$, which implies (iv). □

Proof of Theorem 3.1. We recall that

$$\mathscr{M} = \{v \in H^1(\mathbb{R}^N)\setminus\{0\}; Q(v) = 0, I(v) \leqslant 0\},$$

and define

$$d_{\mathscr{M}} := \inf\{S(v); v \in \mathscr{M}\}.$$

We proceed in three steps.

Step 1. Let us prove $d(\omega) = d_{\mathscr{M}}$. Since the ground states φ satisfy $Q(\varphi) = I(\varphi) = 0$, we have $\varphi \in \mathscr{M}$. Combined with $S(\varphi) = d(\omega)$, this implies $d_{\mathscr{M}} \leqslant d(\omega)$. Conversely, let $v \in \mathscr{M}$. If $I(v) = 0$, then trivially $S(v) \geqslant d(\omega)$, thus we suppose $I(v) < 0$. We use the rescaling defined in Lemma 3.3 : for $\lambda > 0$ we have

$$I(v^\lambda) = \lambda^2\|\nabla v\|_2^2 + \omega\|v\|_2^2 - \int_{\mathbb{R}^N}\lambda^{-N/2}g(\lambda^{N/2}|v|)|v|dx.$$

It follows from (A_0)-(b) that $\lim_{\lambda\to 0} I(v^\lambda) = \omega\|v\|_2^2$ and thus by continuity there exists $\lambda_1 < 1$ such that $I(v^{\lambda_1}) = 0$. Thus $S(v^{\lambda_1}) \geqslant d(\omega)$. Now, from $Q(v) = 0$ and (iii) in Lemma 3.3 we have

$$S(v) \geqslant S(v^{\lambda_1}) \geqslant d(\omega),$$

hence $d_{\mathscr{M}} = d(\omega)$.

Step 2. For $\lambda > 0$, we set $u^\lambda := \varphi^\lambda$. For $\lambda > 1$ close to 1, we have

$$S(u^\lambda) < S(\varphi) \text{ and } Q(u^\lambda) < 0, \tag{3.12}$$

$$I(u^\lambda) < 0. \tag{3.13}$$

Indeed, (3.12) follows from (iii) and (v) in Lemma 3.3. For (3.13), we write

$$\begin{aligned} I(u^\lambda) &= 2S(u^\lambda) + \frac{2}{N}Q(u^\lambda) - \frac{2}{N}\|\nabla u^\lambda\|_2^2 \\ &\leqslant 2S(\varphi) + \frac{2}{N}Q(\varphi) - I(\varphi) - \frac{2\lambda^2}{N}\|\nabla\varphi\|_2^2 \\ &\leqslant \frac{2(1-\lambda^2)}{N}\|\nabla\varphi\|_2^2 < 0. \end{aligned}$$

Let $u(t)$ be the solution of (3.3) with $u(0) = u^\lambda$. We claim that the properties described in (3.12), (3.13) are invariant under the flow of (3.3). Indeed, since from (3.5) we have for all $t > 0$

$$S(u(t)) = S(u^\lambda) < S(\varphi), \tag{3.14}$$

we infer that $I(u(t)) \neq 0$ for any $t \geqslant 0$, and by continuity we have $I(u(t)) < 0$ for all $t \geqslant 0$. It follows that $Q(u(t)) \neq 0$ for any $t \geqslant 0$ (if not $u(t) \in \mathscr{M}$ and thus $S(u(t)) \geqslant S(\varphi)$ which contradicts (3.14)), and by continuity we have $Q(u(t)) < 0$ for all $t \geqslant 0$. Thus for all $t > 0$ we have

$$S(u(t)) < S(\varphi), I(u(t)) < 0 \text{ and } Q(u(t)) < 0.$$

Step 3. We fix $t > 0$ and define $v := u(t)$. For $\beta > 0$, let $v^\beta(x) := \beta^{\frac{N}{2}} v(\beta x)$. From Step 2 we have $Q(v) < 0$, thus from Lemma 3.3 there exists $\beta_0 < 1$ such that $Q(v^{\beta_0}) = 0$. If $I(v^{\beta_0}) \leqslant 0$, we keep β_0, otherwise we replace it by $\tilde{\beta}_0 \in (\beta_0, 1)$ such that $I(v^{\tilde{\beta}_0}) = 0$. Thus in any case we have

$$S(v^{\beta_0}) \geqslant d(\omega) \tag{3.15}$$

and $Q(v^{\beta_0}) \leqslant 0$. Now from (iv) in Lemma 3.3, we have

$$S(v) - S(v^{\beta_0}) \geqslant (1-\beta_0)\frac{\partial}{\partial\beta}S(v^\beta)_{|\beta=1}.$$

Thus, from (v) in Lemma 3.3, $Q(v) < 0$ and $\beta_0 < 1$, we get

$$S(v) - S(v^{\beta_0}) \geqslant Q(v).$$

Combined with (3.15), this gives

$$Q(v) \leqslant S(v) - d(\omega) := -\delta < 0 \tag{3.16}$$

where δ is independent of t since S is a conserved quantity.

To conclude, it suffices to observe that thanks to (3.7) and (3.16) we have

$$\|xu(t)\|_2^2 \leqslant -4\delta t^2 + Ct + \|xu^\lambda\|_2^2, \tag{3.17}$$

and since the right hand side of (3.17) becomes negative when t grows up, we easily deduce that $T_{u^\lambda} < +\infty$ and $\lim_{t\to T_{u^\lambda}} \|\nabla u(t)\|_2 = +\infty$. □

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Chapitre 4

Instability for standing waves of nonlinear Klein-Gordon equations via mountain-pass arguments

Abstract. We introduce mountain-pass type arguments in the context of instability for Klein-Gordon equations. Our aim is to illustrate on two examples how these arguments can be useful to simplify proofs and derive new results of orbital stability/instability. For a power-type nonlinearity, we prove that the ground states of the associated stationary equation are minimizers of the functional *action* on a wide variety of constraints. For a general nonlinearity, we extend to the dimension $N = 2$ the classical instability result for stationary solutions of nonlinear Klein-Gordon equations proved in 1985 by Shatah in dimension $N \geqslant 3$.

4.1 Introduction

The aim of the present paper is to show how recent methods and results concerning the variational characterizations of the ground states for elliptic equations of the form

$$-\Delta\varphi = g(\varphi), \quad \varphi \in H^1(\mathbb{R}^N;\mathbb{C}) \tag{4.1}$$

can be used to study the orbital stability/instability of the standing waves of various nonlinear equations such as Schrödinger equations, Klein-Gordon equations, generalized Boussinesq equations, etc. Our work is motivated by recent developments (see for instance [10, 16, 17, 18, 21, 22]) of the techniques introduced by Berestycki and

Cazenave [2] to prove the instability of standing waves for nonlinear evolution equations. We present our approach on two examples involving nonlinear Klein-Gordon equations of the form

$$u_{tt} - \Delta u + \rho u = f(u) \tag{4.2}$$

where $\rho > 0$, $u : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{C}$ and $f : (0, +\infty) \mapsto \mathbb{R}$ is extended to $\mathbb{C}$ by setting $f(z) = f(|z|)z/|z|$ for $z \in \mathbb{C} \setminus \{0\}$ and $f(0) = 0$.

A standing wave of (4.2) is a solution of the form $e^{i\omega t}\varphi_\omega(x)$ for $\omega \in \mathbb{R}$ and $\varphi_\omega \in H^1(\mathbb{R}^N; \mathbb{C})$. Thus φ_ω satisfies

$$-\Delta\varphi_\omega + (\rho - \omega^2)\varphi_\omega - f(\varphi_\omega) = 0. \tag{4.3}$$

Clearly, (4.3) is of the form (4.1). From now on we write $H^1(\mathbb{R}^N)$ for $H^1(\mathbb{R}^N; \mathbb{C})$. The *least energy level* m is defined by

$$m := \inf\{S(v) \big| v \in H^1(\mathbb{R}^N) \setminus \{0\},\ v \text{ is a solution of (4.1)}\} \tag{4.4}$$

where $S : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ is the natural functional (often called *action*) corresponding to (4.1)

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 - \int_{\mathbb{R}^N} G(v)dx,$$

with $G(s) := \int_0^{|s|} g(t)dt$. A solution $\varphi \in H^1(\mathbb{R}^N)$ of (4.1) is said to be a *ground state*, or *least energy solution*, if

$$S(\varphi) = m.$$

The study of the existence for solutions of (4.1) goes back to the work of Strauss [25] (see also [12]). The most general result in that direction is due to Berestycki and Lions [5] for $N = 1$ and $N \geqslant 3$ and Berestycki, Gallouet and Kavian [3] for $N = 2$.

The assumptions of [3, 5] when $N \geqslant 2$ are :

(g0) g is continuous and odd,

(g1) if $N \geqslant 3$, $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leqslant \limsup_{s\to 0} \frac{g(s)}{s} < 0$,
if $N = 2$, $-\infty < \lim_{s\to 0} \frac{g(s)}{s} := -\rho < 0$,

(g2) if $N \geqslant 3$, $\lim_{s\to+\infty} \frac{g(s)}{s^{\frac{N+2}{N-2}}} = 0$,
if $N = 2$, $\forall \alpha > 0\ \exists C_\alpha > 0$ such that $|g(s)| \leqslant C_\alpha e^{\alpha s^2}\ \forall s > 0$.

(g3) there exists $\xi_0 > 0$ such that $G(\xi_0) > 0$.

It is known that the assumptions (g0)-(g3) are almost optimal to insure the existence of a solution for (4.1) (see [5, Section 2.2]). In [3, 5] it is proved that for $N \geqslant 2$ and under (g0)-(g3) there exists a positive radial least energy solution φ of (4.1) when the infimum in (4.4) is taken over the solutions belonging to $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover it is easily deduce from the proofs in [3, 5] that this φ is still a least energy solution of (4.1) when the infimum is, as in (4.4), taken over the set of all complex valued solutions. See [11] for a proof of this statement along with a description of the ground states as being of the form $U = e^{i\theta}\tilde{U}$ where $\theta \in \mathbb{R}$ and $\tilde{U}$ is a real positive ground state solution of (4.1).

In dimension $N = 1$, the assumptions in [5] are

(h0) g is locally Lipschitz continuous and $g(0) = 0$,

(h1) there exists $\eta_0 > 0$ such that

$$G(s) < 0 \text{ for all } s \in (0, \eta_0),\ G(\eta_0) = 0,\ g(\eta_0) > 0$$

and it is proved in [5] that under (h0) the condition (h1) is necessary and sufficient to guarantee the existence of a unique (up to translation) real positive solution of (4.1). Here also, it can be shown (see [11]) that the least energy levels coincide for complex and real valued solutions of (4.1).

Since the pioneer works [2, 9], it is known that the stability/instability of the standing waves is closely linked to additional variational characterizations that the associated ground states enjoy. Recently, in [13] for $N \geqslant 2$ and in [14] for $N = 1$, Jeanjean and Tanaka showed that, under the conditions (g0)-(g3) for $N \geqslant 2$ and basically (h0)-(h1) for $N = 1$, the functional S admits a mountain pass geometry. Precisely they show that setting

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0,\ S(\gamma(1)) < 0\} \tag{4.5}$$

one has $\Gamma \neq \emptyset$ and

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} S(\gamma(t)) > 0. \tag{4.6}$$

Furthermore, they proved that

$$c = m,$$

namely that the mountain pass value gives the least energy level. In fact, the results of [13, 14] are proved within the space $H^1(\mathbb{R}^N, \mathbb{R})$ but it is straightforward to show, see Lemma 4.14, that this equality also holds in $H^1(\mathbb{R}^N)$.

In this paper, we will show, by studying two specific problems, how the ideas and methods developed in [13, 14] can be implemented in the context of instability by blow-up for nonlinear Klein-Gordon equations.

First, working with a nonlinearity of power type ($f(s) = |s|^{p-1}s$) we find a set of constraints on which the ground states are minimizers of S. In particular, this gives an alternative, much simpler proof of results in [17, 21, 22] concerning the derivation of an additional variational characterization of the ground states. Precisely, we prove

Theorem 4.1. *Let $\alpha, \beta \in \mathbb{R}$ be such that*

$$\begin{cases} \quad \beta < 0, \quad \alpha(p-1) - 2\beta \geqslant 0 \text{ and } 2\alpha - \beta(N-2) > 0 \\ or \;\; \beta \geqslant 0, \quad \alpha(p-1) - 2\beta \geqslant 0 \text{ and } 2\alpha - \beta N > 0. \end{cases} \tag{4.7}$$

Let $\omega \in (-1, 1)$ and $\varphi_\omega \in H^1(\mathbb{R}^N)$ be a ground state solution of

$$-\Delta\varphi_\omega + (1-\omega^2)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0.$$

Then

$$S(\varphi_\omega) = \min\{S(v) \big| v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\alpha,\beta}(v) = 0\}$$

where

$$\begin{aligned} S(v) &:= \frac{1}{2}\|\nabla v\|_2^2 + \frac{1-\omega^2}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}. \\ K_{\alpha,\beta}(v) &:= \tfrac{2\alpha-\beta(N-2)}{2}\|\nabla v\|_2^2 + \tfrac{(2\alpha-\beta N)(1-\omega^2)}{2}\|v\|_2^2 - \tfrac{\alpha(p+1)-\beta N}{p+1}\|v\|_{p+1}^{p+1}. \end{aligned}$$

The functional $K_{\alpha,\beta}$ is based on the rescaling $v_\lambda(\,\cdot\,) := \lambda^\alpha v(\lambda^\beta\,\cdot\,)$ for $v \in H^1(\mathbb{R}^N)$, precisely, $K_{\alpha,\beta}(v) = \frac{\partial}{\partial\lambda}S(v_\lambda)_{|\lambda=1}$. The main idea of the proof of Theorem 4.1 is to use rescaled functions to construct for any $v \in H^1(\mathbb{R}^N)$ such that $K_{\alpha,\beta}(v) = 0$ a path in Γ attaining his maximum at v.

It is also of interest to consider a limit case of Theorem 4.1.

Theorem 4.2. *Let $\alpha, \beta \in \mathbb{R}$ be such that*

$$\begin{cases} \quad \beta < 0, \quad \alpha(p-1) - 2\beta \geqslant 0 \text{ and } 2\alpha - \beta(N-2) = 0 \\ or \;\; \beta > 0, \quad \alpha(p-1) - 2\beta \geqslant 0 \text{ and } 2\alpha - \beta N = 0. \end{cases} \tag{4.8}$$

Let $\omega \in (-1, 1)$ and φ_ω be a ground state solution of

$$-\Delta\varphi_\omega + (1-\omega^2)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0.$$

Then

$$S(\varphi_\omega) = \min\{S(v) \big| v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\alpha,\beta}(v) = 0\}.$$

Remark 4.1. Looking to the proofs of Theorems 4.1 and 4.2 one see that our Theorems remain unchanged when $(1-\omega^2)$ is replaced by any $m > 0$. We choose however to present our results in the setting of [17, 21, 22].

For $(\alpha, \beta) = (\frac{N}{2}, 1)$, Theorem 4.2 gives a simpler proof of a variational characterization of the ground state proved by Berestycki and Cazenave [2] for $1 + \frac{4}{N} < p < 1 + \frac{4}{N-2}$ and by Nawa [19, Proposition 2.5] for $p = 1 + \frac{4}{N}$. This characterization is at the heart of the classical result of Berestycki and Cazenave [2] dealing with the instability of the ground states of nonlinear Schrödinger equations.

For our second direction of application we consider the instability of the stationary solutions of

$$u_{tt} - \Delta u = g(u). \tag{4.9}$$

In 1985, Shatah established in [23] that under the conditions (g0)-(g3) the radial ground states solutions associated with the standing waves corresponding to $\omega = 0$ are unstable when $N \geqslant 3$. Under stronger hypothesis, but in any dimension and for non necessary radial solutions, Berestycki and Cazenave [2] had previously proved that these ground states are unstable by blow up in finite time. In [23], instability may occur by blow up in infinite time, in the sense that the $H^1(\mathbb{R}^N)$-norm of a solution starting close to a ground state goes to infinity when $t \to +\infty$. Here, we show that the same result hold when $N = 2$.

We make the following hypothesis on the existence and properties of solutions for (4.9).

Assumption H. *For all* $(u_0, v_0) \in H^1_{\text{rad}}(\mathbb{R}^2) \times L^2_{\text{rad}}(\mathbb{R}^2)$ *there exist* $0 < T \leqslant +\infty$ *and* $u : [0, T) \times \mathbb{R}^2 \to \mathbb{C}$ *such that*

- $(u(0), u_t(0)) = (u_0, v_0)$,
- u *(resp.* u_t*) is weakly continuous in* $H^1_{\text{rad}}(\mathbb{R}^2)$ *(resp.* $L^2_{\text{rad}}(\mathbb{R}^2)$*),*
- u *satisfies (4.9) in the sense of distributions,*
- $E(u(t), u_t(t)) \leqslant E(u_0, v_0)$ *for all* $t \in [0, T)$ *(*energy inequality*),*
- *if* $T < +\infty$*, there exists* $(t_n) \subset [0, T)$ *such that* $t_n \to T$ *as* $n \to +\infty$ *and* $\lim_{t_n \to T} \|u(t_n)\|_{H^1(\mathbb{R}^2)} = +\infty$ *(*blow-up alternative*),*

The *energy* E is defined for $u \in H^1(\mathbb{R}^N)$ and $v \in L^2(\mathbb{R}^N)$ by

$$E(u, v) := \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^2} G(u)dx.$$

In what follows, as above, we write $H^1_{\text{rad}}(\mathbb{R}^N)$ (resp. $L^2_{\text{rad}}(\mathbb{R}^N)$) for the space of radial functions of $H^1(\mathbb{R}^N)$ (resp. $L^2(\mathbb{R}^N)$).

Remark 4.2. When $N \geqslant 3$, Shatah claims that Assumption H holds under (g0)-(g3) without any additional restrictions. For others dimensions, Assumption H is known to hold under stronger assumptions on g, see, for example, [8, Chapter 6]. From now on a solution of (4.9) with initial data (u_0, v_0) will refer to a solution of (4.9) with initial data (u_0, v_0) as given by Assumption H.

Our third main result is the following

Theorem 4.3. *Assume $N = 2$, (g0)-(g3) and Assumption H. Let φ be a radial ground state of (4.1). Then φ viewed as a stationary solution of (4.9) is strongly unstable. Namely for all $\varepsilon > 0$ there exist $u_\varepsilon \in H^1(\mathbb{R}^2)$, $T_\varepsilon \in (0, +\infty]$ and $(t_n) \subset (0, T_\varepsilon)$ such that $\|\varphi - u_\varepsilon\|_{H^1(\mathbb{R}^2)} < \varepsilon$ and $\lim_{t_n \to T_\varepsilon} \|u(t_n)\|_{H^1(\mathbb{R}^2)} = +\infty$, where $u(t)$ is a solution of (4.9) with initial data $(u_\varepsilon, 0)$.*

It is still an open question to describe what happen in dimension $N = 1$. Indeed, the use of the radial compactness lemma of Strauss (see Lemma 4.5) restricts our proof to dimensions $N \geqslant 2$. A partial answer is given by the work of Berestycki and Cazenave : for nonlinearities satisfying some additional assumptions (see [2, (H.3)]), the stationary solutions are unstable.

We do hope that the methods developed in this paper will find other areas of applications. In that direction, we mention the work [15] in which the variational characterization $c = m$ derived from [13, 14] is essential to get an alternative, more general proof of the classical result of Berestycki and Cazenave [2] on the instability by blow-up for nonlinear Schrödinger equations.

This paper is organized as follows. In Section 4.2 we prove Theorem 4.1 and Theorem 4.2. In Section 4.3 we prove Theorem 4.3. The proof that the results of [13, 14] extend to the complex case along with a technical lemma are given in the Appendix.

4.2 Variational characterizations of the ground states

In this section, we consider (4.3) with a power type nonlinearity :

$$- \Delta\varphi_\omega + (1 - \omega^2)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0 \tag{4.10}$$

where $1 < p < 1 + 4/(N - 2)$ and $|\omega| < 1$. For this nonlinearity it is known (see [7, Section 8.1] and the references therein) that there exists a unique positive radial ground state $\varphi_\omega \in H^1(\mathbb{R}^N,\mathbb{R})$ of (4.10) and that all ground states are of the form $e^{i\theta}\varphi_\omega(\cdot - y)$ for some fixed $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^N$. The standing waves $e^{i\omega t}\varphi_\omega$ are solutions of the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + u = |u|^{p-1}u \tag{4.11}$$

and the natural functional associated with (4.10) becomes

$$S(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1-\omega^2}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}.$$

Various results of instability for the standing waves of (4.11) were recently proved in [17, 21, 22]. For instance, it was proved in [21] that for any $1 < p < 1+4/(N-2)$ the standing wave associated with a ground state of (4.10) is strongly unstable by blow up if $\omega^2 \leqslant (p-1)/(p-3)$ and $N \geqslant 3$. In [22], a result of strong instability was showed for the optimal range of parameter ω in dimension $N \geqslant 2$ (namely $|\omega| < \omega_c$, where ω_c was determined in [24]). In both cases, it is central in the proofs that the ground states can be characterized as minimizers on constraints having all the form

$$\mathscr{K}_{\alpha,\beta} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} \big| K_{\alpha,\beta}(v) = 0\}$$

for some $\alpha, \beta \in \mathbb{R}$. Recall that the functional $K_{\alpha,\beta}$ is defined for $v \in H^1(\mathbb{R}^N)$ by

$$\begin{aligned} K_{\alpha,\beta}(v) &:= \tfrac{\partial}{\partial\lambda} S(\lambda^\alpha v(\lambda^\beta\,\cdot))_{|\lambda=1} \\ &= \tfrac{2\alpha-\beta(N-2)}{2}\|\nabla v\|_2^2 + \tfrac{(2\alpha-\beta N)(1-\omega^2)}{2}\|v\|_2^2 - \tfrac{\alpha(p+1)-\beta N}{p+1}\|v\|_{p+1}^{p+1}. \end{aligned}$$

For example, it is proved in [21] that the ground states are minimizer of S on $\mathscr{K}_{\alpha,\beta}$ for $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (0,-1/N)$ (see [21, (2.1)]) whereas in [22], the values of (α,β) considered are $(\alpha,\beta) = (N/2,1)$ if $p \geqslant 1+4/N$ (see [22, (2.11)]) and $(\alpha,\beta) = (2/(p-1),1)$ if $p < 1+4/N$ (see [22, (2.18)]). Recently, Liu, Ohta and Todorova [17] extended the approach of [21] to the dimensions $N = 1,2$. Once more, a main feature of their proof is to minimize S on $\mathscr{K}_{\alpha,\beta}$, but this time with

$$\alpha = \frac{(p-1)-(p+3)\omega^2}{2(p-1)\omega^2},\ \beta = -1.$$

In [17, 21, 22], the proofs that the ground states are minimizers of S on $\mathscr{K}_{\alpha,\beta}$ follow similar schemes. First, one has to show the convergence of a minimizing sequence to some function solving a Lagrange equation. After that, the difficulty is to get rid of the Lagrange multiplier. For each choice of (α,β), long computations are involved to prove that the Lagrange multiplier is 0 and to conclude that the obtained function is in fact a solution of (4.10).

Our proof of Theorem 4.1 relies on the following lemma. We recall that Γ is defined in (4.5).

Lemma 4.3. *Let $\alpha, \beta \in \mathbb{R}$ satisfy (4.7). Then for all $v \in \mathscr{K}_{\alpha,\beta}$ we can construct a path γ in Γ such that*

$$\max_{t\in[0,1]} S(\gamma(t)) = S(v).$$

Proof. Let $v \in \mathscr{K}_{\alpha,\beta}$. For all $\lambda \in (0,+\infty)$ we define $v_\lambda \in H^1(\mathbb{R}^N)$ by $v_\lambda(\,\cdot\,) := \lambda^\alpha v(\lambda^\beta\,\cdot\,)$. The idea is to construct the path such that $\gamma(t) = v_{Ct}$ for some $C > 0$.

The first thing to check is that we can extend γ at 0 by continuity. Namely, we must show that under (4.7) we have $\lim_{\lambda\to 0} \|v_\lambda\|_{H^1(\mathbb{R}^N)} = 0$. This is immediate if we remark that

$$\|v_\lambda\|^2_{H^1(\mathbb{R}^N)} = \lambda^{2\alpha-\beta(N-2)}\|\nabla v\|_2^2 + \lambda^{2\alpha-\beta N}\|v\|_2^2,$$

and that (4.7) implies

$$2\alpha - \beta(N-2) > 0 \text{ and } 2\alpha - \beta N > 0.$$

The next step is to prove that $\lambda \to S(v_\lambda)$ increases for $\lambda \in (0,1)$, attains its maximum at $\lambda = 1$ and decreases toward $-\infty$ on $(1,+\infty)$. We have

$$S(v_\lambda) = \frac{\lambda^{2\alpha-\beta(N-2)}}{2}\|\nabla v\|_2^2 + \frac{(1-\omega^2)\lambda^{2\alpha-\beta N}}{2}\|v\|_2^2 - \frac{\lambda^{(p+1)\alpha-\beta N}}{p+1}\|v\|_{p+1}^{p+1}$$

and from easy computations it comes

$$\begin{aligned}\lambda^{-(2\alpha-\beta N-1)}\frac{\partial}{\partial\lambda}S(v_\lambda) &= \lambda^{2\beta}\frac{2\alpha-\beta(N-2)}{2}\|\nabla v\|_2^2 + \frac{(2\alpha-\beta N)(1-\omega^2)}{2}\|v\|_2^2 \\ &\quad -\lambda^{\alpha(p-1)}\frac{\alpha(p+1)-\beta N}{p+1}\|v\|_{p+1}^{p+1}.\end{aligned}$$

Therefore, if α and β satisfy

$$\left\{\begin{array}{ll} & \beta \neq 0 \text{ and } \alpha(p-1) \geqslant 2\beta \\ \text{or} & \beta = 0 \text{ and } \alpha(p-1) > 0 \end{array}\right. \tag{4.12}$$

then

$$\left\{\begin{array}{l} \frac{\partial}{\partial\lambda}S(v_\lambda) > 0 \text{ for } \lambda \in (0,1), \\ \frac{\partial}{\partial\lambda}S(v_\lambda) < 0 \text{ for } \lambda \in (1,+\infty), \\ \lim_{\lambda\to+\infty} S(v_\lambda) = -\infty. \end{array}\right.$$

Since $\alpha > 0$ when $\beta = 0$ in (4.7) it is clear that (4.12) hold under (4.7).

Finally, choosing C large enough to have $S(v_C) < 0$ and defining $\gamma : [0,1] \mapsto H^1(\mathbb{R}^N)$ by

$$\gamma(0) := 0 \text{ and } \gamma(t) := v_{tC}$$

we have a path satisfying the conclusion of the lemma. $\square$

Proof of Theorem 4.1. Let φ_ω be a least energy solution of (4.10) for $|\omega| < 1$. From Lemma 4.14 we know that

$$c = m$$

where m is the least energy level and c the mountain pass value (see (4.4) and (4.6) for the definitions of m and c). Since φ_ω is a solution of (4.10), $\varphi_\omega \in \mathcal{C}^1$ and φ_ω, $\nabla\varphi_\omega$ are exponentially decaying at infinity (see, for example, [7, Theorem 8.1.1]); in particular, $x.\nabla\varphi_\omega \in H^1(\mathbb{R}^N)$, and

$$K_{\alpha,\beta}(\varphi_\omega) = \frac{\partial}{\partial\lambda} S(\lambda^\alpha \varphi_\omega(\lambda^\beta \cdot))\big|_{\lambda=1} = \langle S'(\varphi_\omega), \alpha\varphi_\omega + \beta x.\nabla\varphi_\omega\rangle = 0.$$

Thus $\varphi_\omega \in \mathscr{K}_{\alpha,\beta}$ and

$$\min\{S(v)\big|v \in \mathscr{K}_{\alpha,\beta}\} \leqslant S(\varphi_\omega) = c. \tag{4.13}$$

Conversely, it follows from Lemma 4.3 that

$$c \leqslant \min\{S(v)\big|v \in \mathscr{K}_{\alpha,\beta}\}. \tag{4.14}$$

To combine (4.13) and (4.14) finishes the proof. □

We now turn to the proof of Theorem 4.2. It follows the same lines as for Theorem 4.1 : find a path reaching its maximum on the constraint $\mathscr{K}_{\alpha,\beta}$ and use the equality $c = m$. The main difference is in the way we construct the path : we still want to use the rescaled functions v_λ, but their $H^1(\mathbb{R}^N)-$norm does not any more converge to 0 as $\lambda \to 0$. This difficulty is overcome by gluing to $\{v_\lambda\}_{\lambda>\lambda_0}$ a path linking 0 to v_{λ_0} for λ_0 suitably chosen. The lemma is

Lemma 4.4. *Let $\alpha, \beta \in \mathbb{R}$ satisfy (4.8). Then for all $v \in \mathscr{K}_{\alpha,\beta}$ we can construct a path γ in Γ such that*

$$\max_{t\in[0,1]} S(\gamma(t)) = S(v).$$

Proof. Let $v \in \mathscr{K}_{\alpha,\beta}$ and $v_{\lambda_0}(\cdot) := \lambda_0^\alpha v(\lambda_0^\beta \cdot)$ for some $\lambda_0 \in (0, 1)$ whose value will be fixed later. Let $C > 0$ be such that $S(v_C) < 0$ and consider the curves

$$\begin{aligned} \Lambda_1 &:= \{v_\lambda\big|\lambda \in [\lambda_0, C]\}, \\ \Lambda_2 &:= \{tv_{\lambda_0}\big|t \in [0, 1]\}. \end{aligned}$$

To get a path as desired, we will glue the two curves Λ_1 and Λ_2. It is clear that as in the proof of Lemma 4.3, S attained its maximum on Λ_1 at v. Thus the only thing we have to check is that $t \mapsto S(tv_{\lambda_0})$ is increasing on $[0, 1]$.

We have

$$\frac{\partial}{\partial t} S(tv_{\lambda_0}) = t(\|\nabla v_{\lambda_0}\|_2^2 + (1-\omega^2)\|v_{\lambda_0}\|_2^2 - t^{p-1}\|v_{\lambda_0}\|_{p+1}^{p+1}).$$

If $\beta > 0$ and $\alpha = \beta N/2$ (see (4.8)), then $\lambda_0 \to \|v_{\lambda_0}\|_2$ is constant. If $\beta < 0$ and $\alpha = \beta(N-2)/2$ then $\lambda_0 \to \|\nabla v_{\lambda_0}\|_2$ is constant. Moreover, we have in any case

$$\lim_{\lambda_0 \to 0} \|v_{\lambda_0}\|_{p+1}^{p+1} = 0.$$

Therefore, if $\lambda_0 \in (0,1)$ is small enough we have

$$\frac{\partial}{\partial t} S(tv_{\lambda_0}) > 0 \text{ for } t \in (0,1).$$

To define $\gamma : [0,1] \mapsto H^1(\mathbb{R}^N)$ by

$$\begin{cases} \gamma(t) &= \frac{Ct}{\lambda_0} v_{\lambda_0} \text{ for } t \in [0, \frac{\lambda_0}{C}) \\ \gamma(t) &= v_{Ct} \text{ for } t \in [\frac{\lambda_0}{C}, 1] \end{cases}$$

gives us the desired path. □

Proof of Theorem 4.2. The proof is identical to the proof of Theorem 4.1 with Lemma 4.3 replaced by Lemma 4.4. □

4.3 Instability for a generalized nonlinear Klein-Gordon equation

In this section, we consider the nonlinear Klein-Gordon equation with a general nonlinearity

$$u_{tt} - \Delta u = g(u). \tag{4.15}$$

In [23], Shatah proved that for $N \geqslant 3$, under (g0)-(g3), the radial ground states solutions of

$$-\Delta \varphi = g(\varphi), \quad \varphi \in H^1(\mathbb{R}^N) \tag{4.16}$$

viewed as stationary solutions of (4.15) are unstable in the sense of Theorem 4.3.

The restriction to $N \geqslant 3$ has its origin in, at least, two reasons.

First, one needs to control the decay in $|x|$ of $u(t,x)$ uniformly in t. This appears in the proofs of Proposition 4.12 and Lemma 4.15. For this control, the following compactness lemma due to Strauss [25] is used.

Lemma 4.5. *Let $N \geqslant 2$ and $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Then*

$$|v(x)| \leqslant C|x|^{\frac{1-N}{2}} \|v\|_{H^1(\mathbb{R}^N)} \text{ a.e.}$$

with C independent of x and u. In particular, the following injection is compact

$$H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \text{ for } 2 < q < 2^\star,$$

where $2^\star = \frac{2N}{N-2}$ if $N \geqslant 3$ and $2^\star = +\infty$ if $N = 2$.

Actually, to use this lemma only $N \geqslant 2$ is necessary.

A second reason for the restriction $N \geqslant 3$ in [23] is found in the use of a constraint based on Pohozaev's identity to derive a variational characterization of the ground states, to define an invariant set, and, most important, to choose suitable initial data close to the ground states. Thanks to our approach, we arrive on this second point to require only $N \geqslant 2$.

Our proof will make use of the following variational characterization of the ground states.

Lemma 4.6. *Let $\varphi \in H^1(\mathbb{R}^2)$ be a ground state of (4.16). Then*

$$S(\varphi) = m = \min_{v \in \mathscr{P}} S(v) \tag{4.17}$$

where

$$\mathscr{P} := \{v \in H^1(\mathbb{R}^2) \setminus \{0\} \big| P(v) = 0\}$$

with $P(v) := \displaystyle\int_{\mathbb{R}^2} G(v)dx$ for $v \in H^1(\mathbb{R}^2)$.

This lemma was proved in [3] when $v \in H^1(\mathbb{R}^N, \mathbb{R})$. It can trivially be extended to $v \in H^1(\mathbb{R}^N)$, see [11].

Remark 4.7. The functional P is related to the so-called Pohozaev identity (see [5, Proposition 1]) : for $N \geqslant 1$, any solution $v \in H^1(\mathbb{R}^N)$ of (4.16) satisfies

$$\frac{N-2}{2}\|\nabla v\|_2^2 - N\int_{\mathbb{R}^N} G(v)dx = 0.$$

A main feature of the dimension $N = 2$ is that we lose the control on the $L^2(\mathbb{R}^N)-$norm of ∇v.

Remark 4.8. For $N \geqslant 3$, Shatah also showed that the radial ground states are minimizers of S among all non trivial functions satisfying Pohozaev identity (see [23, Proposition 1.5]). His method consists in proving that the minimization problem has a solution and then to eliminate the Lagrange multiplier. In fact, as it is done in [13, Lemma 3.1], a shorter proof can be performed by simply establishing a correspondence with a minimization problem already solved in [5].

The scheme of the proof is the following : first, define a set $\mathcal{I} \subset H^1_{\text{rad}}(\mathbb{R}^2) \times L^2_{\text{rad}}(\mathbb{R}^2)$ such that any solution of (4.15) with initial data in $\mathcal{I}$ stays in $\mathcal{I}$ for all time and blows up, then prove that the ground states can be approximated by functions in $\mathcal{I}$.

Let $\mathcal{I}$ be defined by

$$\mathcal{I} := \{u \in H^1_{\text{rad}}(\mathbb{R}^2) \setminus \{0\}, v \in L^2_{\text{rad}}(\mathbb{R}^2) \big| E(u,v) < m, P(u) > 0\}.$$

We begin by proving an equivalence between two variational problems.

Lemma 4.9. *We have*

$$m = \min_{v \in \mathscr{P}} S(v) = \min\{T(v) \big| v \in H^1(\mathbb{R}^2) \setminus \{0\}, P(v) \geqslant 0\},$$

where $T(v) := \dfrac{1}{2}\|\nabla v\|_2^2$.

Proof. Let $v \in H^1(\mathbb{R}^2)$. If $v \in \mathscr{P}$, then v satisfies $T(v) = S(v)$ and thanks to Lemma 4.6, $T(v) \geqslant m$. Suppose that $P(v) > 0$. For $\lambda > 0$, define $v_\lambda(\,\cdot\,) := \lambda v(\lambda\,\cdot\,)$. We claim that there exists $\lambda_0 < 1$ such that $P(v_{\lambda_0}) = 0$. Indeed, by (g1)-(g2), for all $\alpha > 0$ there exists $C_\alpha > 0$ such that for $s > 0$

$$g(s) \leqslant \frac{-\rho s}{2} + 2s\alpha C_\alpha e^{\alpha s^2}.$$

We recall that $\rho > 0$ is given in (g1) by $\lim_{s\to 0} g(s)s^{-1} = -\rho$. Therefore, for $s > 0$ we have

$$G(s) \leqslant \frac{-\rho s^2}{4} + C_\alpha(e^{\alpha s^2} - 1)$$

and

$$\int_{\mathbb{R}^2} G(v_\lambda) \leqslant \frac{-\rho\|v_\lambda\|_2^2}{4} + C_\alpha \int_{\mathbb{R}^2} (e^{\alpha v_\lambda^2} - 1)dx. \tag{4.18}$$

We remark that $\|v_\lambda\|_2^2 = \|v\|_2^2$ and

$$\int_{\mathbb{R}^2} (e^{\alpha v_\lambda^2} - 1)dx = \lambda^{-2}\int_{\mathbb{R}^2} (e^{\alpha\lambda^2 v^2} - 1)dx.$$

For $\lambda < 1$ we have

$$\lambda^{-2}(e^{\alpha\lambda^2 v^2(x)} - 1) < e^{\alpha v^2(x)} - 1 \text{ for all } x \in \mathbb{R}^2,$$

and by Moser-Trudinger inequality (see [1, Theorem 8.25]) there exists $\alpha > 0$ such that $(e^{\alpha v^2} - 1) \in L^1(\mathbb{R}^2)$. Hence, Lebesgue's Theorem gives

$$\int_{\mathbb{R}^2} (e^{\alpha v_\lambda^2} - 1)dx \to 0 \text{ when } \lambda \to 0.$$

Coming back to (4.18) this means that

$$\int_{\mathbb{R}^2} G(v_\lambda) < 0 \text{ for } \lambda > 0 \text{ small enough,}$$

and by continuity of P this proves the claim.

Now, we have

$$\inf_{u\in\mathscr{P}} S(u) \leqslant S(v_{\lambda_0}) = T(v_{\lambda_0}) = \lambda_0^2 T(v) < T(v),$$

and the lemma is proved. □

Next we prove that the set $\mathcal{I}$ is invariant under the flow of (4.15).

Lemma 4.10. *Let $(u_0, v_0) \in \mathcal{I}$, $0 < T \leqslant +\infty$ and $u(t)$ a solution of (4.15) on $[0, T)$ with initial data (u_0, v_0). Then $(u(t), u_t(t)) \in \mathcal{I}$ for all $t \in [0, T)$.*

Proof. Let

$$t_0 := \inf \Big(\{t \in [0,T) \big| P(u(t)) \leqslant 0\} \cup \{+\infty\} \Big).$$

Assume by contradiction that $t_0 \neq +\infty$ and consider $(t_n) \subset (t_0, T)$ such that $t_n \downarrow t_0$ with $P(u(t_n)) \leqslant 0$. By Assumption H, $u(t_n) \rightharpoonup u(t_0)$ weakly in $H^1(\mathbb{R}^2)$. Thus we have

$$T(u(t_0)) \leqslant \liminf_{n\to+\infty} T(u(t_n)) \leqslant \liminf_{n\to+\infty} \left[T(u(t_n)) - P(u(t_n))\right]. \tag{4.19}$$

Moreover

$$\liminf_{n\to+\infty} \left[T(u(t_n)) - P(u(t_n))\right] = \liminf_{n\to+\infty} S(u(t_n)) \leqslant \liminf_{n\to+\infty} E(u(t_n), u_t(t_n)) \tag{4.20}$$

and by the energy inequality in Assumption H we get

$$\liminf_{n\to+\infty} E(u(t_n), u_t(t_n)) \leqslant E(u_0, v_0). \tag{4.21}$$

Recalling that $(u_0, v_0) \in \mathcal{I}$, we have

$$E(u_0, v_0) < m. \tag{4.22}$$

Combining (4.19)-(4.22) gives

$$T(u(t_0)) < m. \tag{4.23}$$

Now, take $(\widetilde{t_n}) \subset (0, t_0)$ such that $\widetilde{t_n} \uparrow t_0$. By Lemma 4.16, $v \to P(v)$ is upper weakly semi-continuous, thus

$$P(u(t_0)) \geqslant \limsup_{n\to+\infty} P(u(\widetilde{t_n})) \geqslant 0. \tag{4.24}$$

Now together (4.23) and (4.24) lead to a contradiction with Lemma 4.9. □

The following lemma is a key step in the proof.

Lemma 4.11. *Let $(u_0, v_0) \in \mathcal{I}$ and $u(t)$ an associated solution of (4.15) in $[0, T)$. Then there exists $\delta > 0$ such that $P(u(t)) > \delta$ for all $t \in [0, T)$.*

Proof. Indeed, assume by contradiction that there exists a sequence (t_n) such that $P(u(t_n)) \to 0$ as $n \to +\infty$. Then

$$\begin{aligned} T(u(t_n)) &= S(u(t_n)) + P(u(t_n)) \\ &\leqslant E(u(t_n), u_t(t_n)) + P(u(t_n)). \end{aligned}$$

By the energy inequality in Assumption H this implies

$$T(u(t_n)) \leqslant E(u_0, v_0) + P(u(t_n))$$

and thus

$$T(u(t_n)) < m + P(u(t_n)) - \nu \tag{4.25}$$

with $\nu := m - E(u_0, v_0) > 0$ since $(u_0, v_0) \in \mathcal{I}$. For n large enough we have

$$0 \leqslant P(u(t_n)) < \nu/2$$

and thus (4.25) gives

$$T(u(t_n)) < m - \frac{\nu}{2},$$

which contradicts the result of Lemma 4.9. □

The proof of Theorem 4.3 relies on the following proposition.

Proposition 4.12. *Let $(u_0, v_0) \in \mathcal{I}$ and $u(t)$ an associated solution of (4.15) on $[0, T)$. Then there exists $(t_n) \subset (0, T)$ such that $\lim_{t_n \to T} \|u(t_n)\|_{H^1(\mathbb{R}^2)} = +\infty$,*

Proof. The proof of Proposition 4.12 is similar to the proof of Theorem 2.3 in [23], thus we just indicate the main steps. First, if $T < +\infty$, the assertion of Proposition 4.12 is just the blow up alternative in Assumption H. Thus we suppose $T = +\infty$ and, by contradiction, $(\|u(t)\|_{H^1(\mathbb{R}^N)})$ bounded. Following the line of the proof of Theorem 2.3 in [23], it is not hard to see that there exists $0 < \eta < \delta$ (where δ is given by Lemma 4.11) such that

$$2 \int_{\mathbb{R}^2} G(u)\, dx - \eta \leqslant -\frac{\partial}{\partial t} \mathrm{Re} \int_{\mathbb{R}^2} \theta(t, x) \overline{u_t} x . \nabla u dx \tag{4.26}$$

where $\theta : [0, +\infty) \times \mathbb{R}^2 \mapsto \mathbb{R}$ is such that

$$|\theta(t, x)| \leqslant Ct / \ln(t) \tag{4.27}$$

for all $(t, x) \in [0, +\infty) \times \mathbb{R}^2$. To combine (4.26) and Lemma 4.11 gives

$$\delta \leqslant -\frac{\partial}{\partial t}\mathrm{Re}\int_{\mathbb{R}^2}\theta(t,x)\overline{u_t}x.\nabla u dx. \tag{4.28}$$

Hence, by integrating (4.28) we find

$$\delta t \leqslant -\mathrm{Re}\int_{\mathbb{R}^2}\theta(t,x)\overline{u_t}x.\nabla u dx + \mathrm{Re}\int_{\mathbb{R}^2}\theta(0,x)\overline{v_0}\ x.\nabla u_0 dx. \tag{4.29}$$

Now, by (4.27) and (4.29) there exists $C > 0$ such that

$$\ln(t)\delta \leqslant C(1 + \|\nabla u(t)\|_2\|u_t(t)\|_2). \tag{4.30}$$

But, thanks to the energy inequality $\|u_t(t)\|_2$ is bounded, and $\|\nabla u(t)\|_2$ is bounded by assumption, therefore, for t large enough we reach a contradiction in (4.30). □

In dimension $N \geqslant 3$, it is easily seen that for $\lambda < 1$ the dilatation of a ground state $\varphi_\lambda(\cdot) := \varphi(\frac{\cdot}{\lambda})$ gives a sequence of initial data in $\mathcal{I}$ converging to this ground state. This property, combined with the equivalent of Proposition 4.12, gives immediately the instability of the ground states in [23]. This is not the case any more in dimension $N = 2$ where the dilatation $\varphi_\lambda(\cdot) := \varphi(\frac{\cdot}{\lambda})$ leaves $\mathscr{P}$ and T invariant. To overcome this difficulty, we borrow and adapt an idea of [6, Proposition 2] which consists in using separately (and successively) a dilatation and a rescaling to get initial data in $\mathcal{I}$ close to the ground states.

Lemma 4.13. *Let $\varphi \in H^1(\mathbb{R}^2)$ be a ground state of (4.16). For all $\varepsilon > 0$ there exists φ_ε such that*

$$\|\varphi - \varphi_\varepsilon\|_{H^1(\mathbb{R}^2)} < \varepsilon,\ S(\varphi_\varepsilon) < S(\varphi),\ P(\varphi_\varepsilon) > 0.$$

Proof. For $\lambda, \mu > 0$ consider $\varphi_{\lambda,\mu}(\cdot) := \lambda\varphi(\frac{\cdot}{\mu})$. Then

$$\frac{\partial}{\partial\lambda}S(\varphi_{\lambda,\mu}) = \lambda^2\|\nabla\varphi\|_2^2 - \mu^2\int_{\mathbb{R}^2} g(\lambda\varphi)\overline{\varphi}dx.$$

To multiply (4.16) by $\overline{\varphi}$ and integrate gives us

$$\|\nabla\varphi\|_2^2 = \int_{\mathbb{R}^2} g(\varphi)\overline{\varphi}dx.$$

Hence, for $\lambda = 1$ we get

$$\frac{\partial}{\partial\lambda}S(\varphi_{\lambda,\mu})\big|_{\lambda=1} = (1-\mu^2)\|\nabla\varphi\|_2^2.$$

Thus, for all $\mu > 1$, there exists $\lambda_\mu > 0$ such that

$$\frac{\partial}{\partial\lambda} S(\varphi_{\lambda,\mu}) < 0 \text{ for } \lambda \in (1-\lambda_\mu, 1+\lambda_\mu)$$

and therefore

$$S(\varphi_{\lambda,\mu}) < S(\varphi) \text{ for } \lambda \in (1, 1+\lambda_\mu). \tag{4.31}$$

Now,

$$\frac{\partial}{\partial\lambda} P(\varphi_{\lambda,\mu})_{\lambda=1} = \mu^2 \int_{\mathbb{R}^2} g(\varphi)\overline{\varphi} dx = \mu^2 \|\nabla\varphi\|_2^2 > 0.$$

Thus, for all $\mu > 0$, there exists Λ_μ such that

$$\frac{\partial}{\partial\lambda} P(\varphi_{\lambda,\mu}) > 0 \text{ for } \lambda \in (1-\Lambda_\mu, 1+\Lambda_\mu)$$

and therefore

$$P(\varphi_{\lambda,\mu}) > 0 \text{ for } \lambda \in (1, 1+\Lambda_\mu). \tag{4.32}$$

Finally, from (4.31)-(4.32), for $\lambda, \mu > 1$ close enough to 1 we get the desired result. □

Proof of Theorem 4.3. Let $\varepsilon > 0$ and φ_ε given in Lemma 4.13. Then $(\varphi_\varepsilon, 0)$ satisfies

$$E(\varphi_\varepsilon, 0) = S(\varphi_\varepsilon) < m \text{ and } P(\varphi_\varepsilon) > 0,$$

namely $(\varphi_\varepsilon, 0) \in \mathcal{I}$. Theorem 4.3 follows now from Proposition 4.12. □

4.4 Appendix

Lemma 4.14. *Let m denote the least energy level defined in (4.4) and c the mountain pass level defined in (4.6). Then $m = c$.*

Proof. In [13, Theorem 0.2] for $N \geqslant 2$ and [14, Theorem 1.2] for $N = 1$ it is shown that when the class Γ is replaced by

$$\tilde{\Gamma} := \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N, \mathbb{R})), \gamma(0) = 0, \, S(\gamma(1)) < 0\}$$

one has

$$\tilde{c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} S(\gamma(t)) = \tilde{m}$$

where $\tilde{m}$ is the least energy level among real valued solutions of (4.1). From [3, 5, 11] we know that $\tilde{m} = m$. Also trivially $c \leqslant \tilde{c}$. Now for each $\gamma \in \Gamma$ we observe that setting $\tilde{\gamma}(t) = |\gamma(t)|$ one has

$$||\nabla\tilde{\gamma}(t)||_2^2 \leqslant ||\nabla\gamma(t)||_2^2 \quad \text{and} \int_{\mathbb{R}^N} G(\tilde{\gamma}(t))dx = \int_{\mathbb{R}^N} G(\gamma(t))dx.$$

Thus $\tilde{\gamma} \in \tilde{\Gamma}$ and $S(\tilde{\gamma}) \leqslant S(\gamma)$. This show that $\tilde{c} \leqslant c$ and ends the proof. □

Now we prove the upper weakly semicontinuity of P. We begin by a convergence lemma

Lemma 4.15. *Let $H \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ be such that*

(H1) For all $\alpha > 0$ there exists $C_\alpha > 0$ such that $|H(s)| \leqslant C_\alpha(e^{\alpha s^2} - 1)$ for all $s \geqslant 1$,

(H2) $H(s) = o(s^2)$ when $s \to 0$.

Let $(u_n) \subset H^1_{\text{rad}}(\mathbb{R}^2)$ be a sequence bounded in $H^1(\mathbb{R}^2)$ such that $u_n \to u$ a.e. Then we have

$$H(u_n) \to H(u) \text{ in } L^1(\mathbb{R}^2).$$

This lemma was proved in [4, Lemma 5.2], the extended version of [3]. We recall it here for the sake of completeness.

Proof of Lemma 4.15. From the continuity of H we have $H(u_n) \to H(u)$ a.e. By a theorem of Vitali (see, for example, [20, p 380]), it is enough to prove

(i) for each $\varepsilon > 0$ there exists $R > 0$ such that $\int_{\mathbb{R}^2\setminus\{|x|<R\}} H(u_n)dx < \varepsilon$ for all $n \in \mathbb{N}$,

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_{\{|x-y|<\delta\}} H(u_n)dx < \varepsilon$ for all $y \in \{x \in \mathbb{R}^2 \text{ such that } |x| < R\}$ (*equiintegrability*).

Let $\varepsilon > 0$ be arbitrary chosen. From (H1)-(H2), for $\alpha > 0$ there exists $C_\alpha > 0$ such that for all $s \in \mathbb{R}$

$$|H(s)| \leqslant \alpha s^2 + C_\alpha(e^{s^2} - 1).$$

Thus, for any $R > 0$

$$\int_{\{|x|>R\}} |H(u_n)| \leqslant \alpha\|u_n\|_2^2 + C_\alpha \int_{\{|x|>R\}} (e^{u_n^2} - 1)dx.$$

On one hand, since (u_n) is bounded in $L^2(\mathbb{R}^N)$ we can take $\alpha > 0$ small enough such that

$$\alpha \|u_n\|_2^2 < \frac{\varepsilon}{2}.$$

On the other hand, from Lemma 4.5 there exists C such that

$$C_\alpha \int_{\{|x|>R\}} (e^{u_n^2} - 1)dx \leqslant C_\alpha \int_{\{|x|>R\}} (e^{C|x|^{-1}} - 1)dx$$

and for $R > 0$ chosen large enough we have

$$C_\alpha \int_{\{|x|>R\}} (e^{C|x|^{-1}} - 1)dx < \frac{\varepsilon}{2}.$$

Therefore, (i) is satisfied.

For (ii), we first remark that, by (H1) and Moser-Trudinger inequality, there exists $\alpha > 0$ and $M > 0$ such that

$$\int_{\{|x|<R\}} H(u_n)dx \leqslant \int_{\{|x|<R\}} e^{\alpha u_n^2} dx < M \text{ for all } n \in \mathbb{N}$$

In particular, then $H(u_n)$ is bounded in $L^r(|x| < R)$ for any $1 < r < +\infty$. Hence (ii) holds by de La Vallée Poussin equiintegrability lemma. □

Lemma 4.16. *The functional $P(v) = \int_{\mathbb{R}^N} G(v)dx$ is of class $\mathcal{C}^1$ and upper weakly semi-continuous in $H^1(\mathbb{R}^N)$.*

Proof. It is standard to show that under (g2), $P \in \mathcal{C}^1(H^1(\mathbb{R}^N), \mathbb{R})$. Now let $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Using (g1)-(g2), we can decompose G in

$$G(s) = -\rho s^2 + H(s)$$

where H satisfies the hypothesis of Lemma 4.15. Hence

$$\int_{\mathbb{R}^N} H(v_n)dx \to \int_{\mathbb{R}^N} H(v)dx \text{ when } n \to +\infty.$$

Since $v \to -\|v\|_2$ is upper weakly semicontinuous, this conclude the proof. □

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Résumé

Cette thèse porte sur l'étude des ondes stationnaires d'équations dispersives non linéaires, en particulier l'équation de Schrödinger, mais aussi celle de Klein-Gordon. Les travaux présentés s'articulent autour de deux questions principales : l'existence et la stabilité orbitale de ces ondes stationnaires.

L'existence est étudiée par des méthodes essentiellement variationnelles. En plus de la simple existence, on met en évidence différentes caractérisations variationnelles des ondes stationnaires, par exemple en tant que points critiques d'une certaine fonctionnelle au niveau du col ou au niveau de moindre énergie, ou encore en tant que minimiseurs d'une fonctionnelle sur différentes contraintes.

Selon la puissance de la non-linéarité et la forme de la dépendance en espace, on démontre que les ondes stationnaires sont stables ou instables. Lorsqu'elles sont instables, on met en évidence que dans certaines situations l'instabilité se manifeste par explosion, tandis que dans d'autres les solutions sont globalement bien posées. En plus des différentes caractérisations variationnelles des ondes stationnaires, les preuves des résultats de stabilité et d'instabilité nécessitent de dériver des informations de nature spectrale. En particulier, dans la première partie de cette thèse, on prouve un résultat de non-dégénérescence du linéarisé pour un problème limite. Dans la deuxième partie, on localise la deuxième valeur propre du linéarisé par la combinaison d'une méthode perturbative et d'arguments de continuation.

Mots clés : ondes stationnaires, stabilité orbitale, instabilité, instabilité par explosion, existence pour les problèmes elliptiques, méthodes variationnelles, arguments de perturbation, méthodes spectrales, équation de Schrödinger non linéaire, équation de Klein-Gordon non linéaire

Abstract

This thesis is devoted to the study of standing waves for nonlinear dispersive equations, in particular the Schrödinger equation but also the Klein-Gordon equation. The works are organized around two main issues : existence and orbital stability of standing waves.

The existence is essentially studied by the way of variational methods. We exhibit various variational characterizations of standing waves, for example as critical points of some functional at the mountain pass level or at the least energy level, or as minimizers of a functional under various constraints.

Depending on the strength of the nonlinearity and on the space dependency, we prove that stability or instability holds for the standing waves. When instability holds, we show that, in some situations, instability occurs by blow up, whereas in other cases the solutions are globally well-posed. In addition to the variational characterization of waves, the study of stability leads us to derive spectral informations. In the first part of this thesis, we show a nondegenerescence result for the linearized operator associated with a limit problem. In the second part, we localize the second eigenvalue of the linearized by the mean of a combinaison of perturbation and continuation arguments.

Keywords : standing waves, orbital stability, instability, instability by blow up, existence for elliptic problems, variational methods, perturbation arguments, spectral theory, nonlinear Schrödinger equation, nonlinear Klein-Gordon equation

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